ADVANCED CALCULUS/LINEAR ALGEBRA BASIC EXAM

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Insructions: Do all 7 problems. Show your work. The passing standards are:

- Master's level: 60% with three questions essentially, complete (including one question from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

Linear Algebra

1. Find a basis for the kernel and image (range) of the following matrix

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 2 & 1 & 4 & 3 \\ 3 & 4 & 1 & 2 \end{pmatrix}.$$

2. Consider the matrix $A = \begin{pmatrix} 1 & 0 & 4 \\ 2 & 3 & 2 \\ 4 & 0 & 1 \end{pmatrix}$.
   (a) Show that A is diagonalizable.
   (b) Find the eigenvalues of A.
   (c) Find an orthogonal basis of eigenvectors of A with respect to the usual inner product.

3. Let V be a finite dimensional vector space and $T : V \to V$ be a linear transformation.
   (a) Show that $\mathrm{rank}(T) \geq \mathrm{rank}(T^2)$.
   (b) If $\mathrm{rank}(T) = \mathrm{rank}(T^2)$ show that $\ker(T) \cap \mathrm{image}(T) = \{\mathbf{0}\}$. Conclude that $V = \ker(T) \oplus \mathrm{image}(T)$ ($\oplus$ denotes direct sum).
   (c) Prove that there is a positive integer m such that $V = \ker(T^m) \oplus \mathrm{rank}(T^m)$.

Advanced Calculus

4. Find the first five coefficients of the Maclaurin series (Taylor series at $x = 0$) of

$$\frac{e^x}{\cos x} = \_\_\_\_\_ + \_\_\_\_\_ x + \_\_\_\_\_ \frac{x^2}{2!} + \_\_\_\_\_ \frac{x^3}{3!} + \_\_\_\_\_ \frac{x^4}{4!}.$$

5. Consider the sequence $a_1, a_2, \ldots$ defined recursively by

$$a_1 = 1, \qquad a_n = 1 + \frac{1}{a_{n-1}}.$$

Give a careful proof that the sequence converges and determine its limit.

6. Find the global extreme values of $f(x, y) = xy$ in the domain $D = \{(x, y) \mid x^2/8 + y^2/2 \le 1\}$.

7. Evaluate

$$\oiint_S \langle x, y, 2 - 2z \rangle \cdot d\mathbf{A},$$

where S is the surface formed by the paraboloid $z = 1 - x^2 - y^2$ above the xy-plane (i.e. $z \ge 0$), with outward pointing normal vector.