DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
JANUARY 2017

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correct.

PhD: 75% with at least three substantially correct.

1. Use Newton's method to find the root 2 of the function

$$f(x) = x^3 - (2a+2)x^2 + (a^2+4a)x - 2a^2 = (x-2)(x-a)^2.$$

Suppose the initial guess is sufficiently close to $x = 2$.

   (a) For which values of a, does Newton's method have only the first order convergence? Compute the convergence rate.

   (b) For what values of a, does Newton's method have the second order convergence?

2. $f(x)$ is a piecewise function defined as

$$f(x) = \begin{cases} \sin x, & x \in [0, x_0], \\ \cos x, & x \in (x_0, 2\pi]. \end{cases}$$

Show that the maximum error for polynomial interpolation does NOT converge when increasing the degree of the polynomials – by showing that there is a positive lower bound for the maximum errors.

3. Find the values of a and b which solve the following optimization problem:

$$\min_{a,b} \int_0^\infty (x^2 - ax - b)^2 \, e^{-x} \, dx.$$

Note that the function $f(x) = (ax+b)$ is the weighted L^2 projection of x^2 onto the space spanned by $\{1, x\}$.

4. Determine the exact conditions on the coefficients a, b, c, d, e under which the following function is a cubic spline:

$$f(x) = \begin{cases} a(x-2)^2 + b(x-1)^3, & \text{if } x \in (-\infty, 1] \\ c(x-2)^2, & \text{if } x \in (1, 3] \\ d(x-3) + e(x-2)^2, & \text{if } x \in (3, \infty) \end{cases}$$

5. Consider the numerical integration rule

$$I(f) = \int_{-h}^{h} f(x)\, dx \approx A_0 f(-h/2) + A_1 f(0) + A_2 f(h/2).$$

   (a) Find A_0, A_1, and A_2 such that the integration rule is exact for polynomials of degree ≤ 2.

   (b) Show that the rule constructed in (a) is in fact exact for polynomials of degree ≤ 3.

   (c) For the constructed rule, it can be proved that

$$I(f) - [A_0 f(-h/2) + A_1 f(0 + A_2 f(h/2)] = c_0 f^{(4)}(\eta) h^5, \ \eta \in (-h, h)$$

   where c_0 is a constant independent of f. Find the constant c_0.

6. For $\dfrac{dw}{dt} = f(t, w)$, consider a multi-step method of the form

$$w_{k+1} = w_k + h(Bf(t_k, w_k) + Cf(t_{k-1}, w_{k-1})),$$

where $h > 0$ is the step-size and $t_k = kh$ for all k.

(a) Define what is meant by the local truncation error for the multi-step method.

(b) Find B and C so that the method is of the highest possible order.

(c) What is the local truncation error for this method?

7. Let A be a real $n \times n$ non-singular upper triangular banded matrix with band width $p > 0$. In other words let a_{ij} be the (i, j) entry of A such that $a_{ij} = 0$ if $i > j$ or $j - i > p$.

(a) Sketch an efficient algorithm that solves the linear system $Ax = b$.

(b) Count the number of Additions/Subtractions and Multiplications/Divisions.