DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST

ADVANCED CALCULUS/LINEAR ALGEBRA EXAM

JANUARY 2017

Do all 7 problems. **Show your work.**

Passing Standard:

- M.S. level: 60% with three questions essentially complete (including at least one from each part);
- Ph.D. level: 75% with two questions from each part essentially complete.

1. Linear Algebra

1.

(1) Let A, B be $n \times n$ matrices. If $AB = 0$ show that

$$\operatorname{rank}(A) + \operatorname{rank}(B) \leq n.$$

(2) For any $n \times n$ matrix A show that there exists a $n \times n$ matrix B with $AB = 0$ and

$$\operatorname{rank}(A) + \operatorname{rank}(B) = n.$$

2. Find the Jordan canonical form of the matrix

$$\begin{pmatrix} 11 & -4 & -5 \\ 21 & -8 & -11 \\ 3 & -1 & 0 \end{pmatrix}.$$

(Hint: the characteristic polynomial of this matrix is $x^3 - 3x^2 + 4$.)

3.

(1) Let T, U be linear transformations of a finite dimensional complex vector space V of dimension n. Show that if $TU = UT$ then T and U have a common eigenvector.
(2) Give an example of linear transformations T, U on a two dimensional *real* vector space V which satisfy $TU = UT$ yet do *not* have a common eigenvector in V.

2. Advanced Calculus

4. Let

$$f(x) = x^2 \int_0^x \cos(t^3)dt.$$

Compute the derivatives $f^{(15)}(0)$ and $f^{(20)}(0)$.

5. For each integer $n \geq 1$, let $f_n(x) = n^2 x^n (1 - x)$.

(1) Show that this sequence of functions converges pointwise on $[0, 1]$.
(2) Does this sequence of functions converges uniformly on $[0, 1]$?
(3) Does

$$\lim_{n\to\infty} \int_0^1 f_n(x)dx = \int_0^1 \lim_{n\to\infty} f_n(x)dx?$$

6. Let $a_0 = 1$ and consider the sequence defined recursively by: $a_n = \log(1 + a_{n-1})$.

(1) Show that $\lim\limits_{n\to\infty} a_n = 0$.
(2) Determine the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n x^n$.

7. Evaluate

$$\oiint_S \langle y, x, z^2 \rangle \cdot d\mathbf{A}$$

with S the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 1$, with outward pointing normal vector.