NAME:

Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.

2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.

3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.

4. State explicitly all results that you use in your proofs and verify that these results apply.

5. Please write your work and answers clearly in the blank space under each question and on the blank page after each question.

1. Let f_n and g_n be sequences in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ respectively such that $1/p + 1/q = 1$ and for some f, g with $f \in L^p(\mathbb{R}^d), g \in L^q(\mathbb{R}^d)$ we have

$$f_n \to f \text{ in } L^p$$
$$g_n \to g \text{ in } L^q$$

Show that the sequence $h_n := f_n g_n$ converges in L^1 to $h := fg$. That is, show that

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} |f_n g_n - fg| \, dx = 0.$$

2. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a continuous function with compact support. Show that

$$\lim_{n\to\infty} \int_{\{(x,y)|\ x\geq 0,\ y\geq 0\}} \frac{n}{2\pi} f(x,y) e^{-n\frac{x^2}{2} - n\frac{y^2}{2}}\ dxdy = \frac{1}{4} f(0,0).$$

Hint: Think first about what limit you would get if above you changed the area of integration from just the upper right quadrant to all of $\mathbb{R}^2$.

Hint: For your convenience, remember that the definite integral $\int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\ dt$ *is equal to* $\sqrt{2\pi}$.

3. Fix f such that $f \in L^1(0,\infty)$ and $f \in L^2(0,\infty)$, then

   (a) Show that $f \in L^r(0,\infty)$ for every $r \in (1,2)$.

   (b) Show that the function

$$\phi(r) = \|f\|_{L^r(0,1)}, \quad r \in [1,2].$$

   is a continuous function of r.
   Hint: Show first that $g(x) = \max\{f(x), f(x)^2\}$ *is an integrable function in* $(0,\infty)$*, or use Hölder inequality for properly chosen exponents...*

4. Let (X, Σ, μ) be a measure space

   (a) State the definition of weak convergence for a sequence in the (real) Hilbert space $L^2(X, \mu)$.

   (b) Let $\{f_n\}_{n\in\mathbb{N}}$ be a family of measurable real valued functions such that (below, δ_{nm} denotes the Kronecker delta)

   $$\int_X f_n(x) f_m(x)\ d\mu(x) = \delta_{nm} \ \ \forall\, n, m \in \mathbb{N}.$$

   Let $g \in L^2(X, \mu)$, is it true that

   $$\sum_n \left(\int g(x) f_n(x)\ d\mu(x) \right)^2 \leq \|g\|^2_{L^2(X,\mu)} \ \ ?.$$

   Prove or give a counter-example.

   (c) Let $\{f_n\}_{n\in\mathbb{N}}$ be as above. Is it true that if g is such that $\int g(x) f_n(x)\ d\mu(x) = 0$ for all $n \in \mathbb{N}$ then $g = 0$? Prove or give a counter-example.

   (d) Let $\{f_n\}_{n\in\mathbb{N}}$ be as before. Show that the sequence $\{f_n\}_{n\in\mathbb{N}}$ must converge to 0 in the weak sense.

5. Consider a sequence of measurable functions $g_n(x)$ on $\mathbb{R}$ such that $\lim_{n\to\infty} g_n(x)$ exists for almost every $x \in \mathbb{R}$. Suppose further that for some $p \in [1, \infty)$ we have

$$G(x) := \sup_n |g_n(x)|^p \in L^1(\mathbb{R}).$$

Then, prove that $g_n \to g$ in the $L^p(\mathbb{R})$ norm.

6. Let (X, Σ, μ) be a measurable space and let $f : X \to \mathbb{R}$ be a measurable function such that

$$\mu(\{n \leq |f| \leq n+1\}) \leq \frac{M}{2^n}$$

Show that for any $0 < p < \infty$, we have

$$\int_X |f|^p \, d\mu(x) < \infty$$

7. Consider a set X with a σ-algebra Σ and a family m_n of measures with respect to Σ. Suppose that

$$\sup_n m_n(X) < \infty.$$

Define a new measre m by setting, for each $E \in \Sigma$

$$m(E) := \sum_n \frac{1}{n^2} m_n(E)$$

Show m is also a measure in Σ (please list the defining properties of a measure), and that each m_n is absolutely continuous with respect to m.

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a function which is locally integrable. Define the **maximal function of** f by

$$f^*(x) := \sup_{I \ni x} \frac{1}{|I|} \int_I |f(y)| \, dy.$$

Here the supremum is taken over all intervals containing x. Suppose that f is not almost-everywhere equal to zero, that is: $\{|f| > \delta\}$ has positive measure for some δ.

(a) Show that if $f \in L^\infty(\mathbb{R})$, then $\|f^*\|_{L^\infty(\mathbb{R})} \leq \|f\|_{L^\infty(\mathbb{R})}$.

(b) Show there exists some $c > 0$ such that

$$f^*(x) \geq \frac{c}{|x|} \text{ whenever } |x| \geq 1.$$

(c) Conclude that the maximal function of $f \in L^1_{loc}(\mathbb{R})$ belongs to $L^1(\mathbb{R})$ only if f is zero a.e.

Hint: For b) note that there must be some $R > 0$ *such that* $\{|f| > \delta\} \cap [-R, R]$ *has positive measure. What does this say about an interval* I *containing* $[-R, R]$*?*