Complex analysis qualifying exam

Department of Mathematics and Statistics
University of Massachusetts, Amherst

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Do 8 out of the following 10 questions.

Each question is worth 10 points. To pass at the Master’s level it is sufficient to have 45 points with 3 questions essentially correct. To pass at the PhD level it is sufficient to have 55 points with 4 questions essentially correct.

Note: All answers should be justified carefully.

(1) Let $C$ be a square with vertices $(\pm 4, \pm 4)$ oriented counter-clockwise. Evaluate the integral

$$\int_{C} \frac{z^5 e^{\frac{1}{z}}}{1 - z^4} dz$$

(2) Find all holomorphic functions $f(z)$ which map the open set

$$\{ z : -\pi < \text{Re}z < \pi \}$$

one-to-one onto the open unit disk

$$\{ z : |z| < 1 \}$$

and such that $f(0) = 0$. Justify your answer!

(3) Let $p_1, p_2, p_3, p_4 \in \mathbb{C}$ be 4 arbitrary distinct points. Let $C$ (resp. $D$) be a unique circle or line passing through points $p_1, p_2, p_3$ (resp. $p_1, p_3, p_4$).

Show that $C$ and $D$ are perpendicular at $p_1$ if and only if the cross-ratio $(p_1, p_2, p_3, p_4)$ is a purely imaginary number.
(4) Let $f(z) = (z - \alpha_1) \ldots (z - \alpha_n)$ be a complex polynomial with derivative $f'(z) = n(z - \beta_1) \ldots (z - \beta_{n-1})$.

(a) Show that if $\Re \alpha_i > 0$ for every $i$ then $\Re \beta_i > 0$ for every $i$ (Hint: consider $f'(\beta_i)/f(\beta_i)$).

(b) Show that if $|\alpha_i| < 1$ for every $i$ then $|\beta_i| < 1$ for every $i$.

(5) Show that the series

$$f(z) = \sum_{n=-\infty}^{\infty} e^{-\pi n^2 + 2\pi i n z}$$

defines an entire function.

(6) Evaluate the improper integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{x^2 + a^2} dx$$

for any positive real number $a$.

(7) Let $\lambda$ be a real number larger than 1. Show that the equation

$$\lambda - z - e^{-z} = 0$$

has precisely one solution in the half plane $\{z : \Re(z) > 0\}$. Moreover, the solution is real.

(8) Let $f$ be a non-constant entire function. Show that the image of $f$ is dense in the complex plane.

(9) Evaluate the integral

$$\int_{0}^{2\pi} \frac{1}{2 - \sin(\theta)} d\theta.$$ 

(10) Consider the Laurent series $\tan(z) = \sum_{n=-\infty}^{\infty} a_n z^n$, which is valid in the annulus $\frac{\pi}{2} < |z| < \frac{3\pi}{2}$. Find the coefficients $a_n$ with index $-\infty < n \leq -1$. Hint: Use the integral formula for the coefficients.