• Do 7 of the following 9 problems. **Show your work!**
• Passing Standard:
  – Master’s level: 60% with three questions essentially complete (including at least one from each part)
  – Ph. D. level: 75% with two questions from each part essentially complete

### Part I. Linear Algebra

1. Find two $2 \times 2$ matrices $A, B$ with $A \neq \pm B$ such that
   $$A^2 = B^2 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}. $$
   **Note:** It is possible to solve this problem *without* doing excessive, brute force computation.

2. Let $A$ be a real, square matrix. Suppose $A = -A^t$.
   (a) Show that if $A$ is $3 \times 3$ then $\det(A) = 0$.
   (b) Show that if $A$ is $4 \times 4$ and invertible then $\det(A)$ is positive.

3. Let $V$ be the vector space of real polynomials of degree $\leq 2$. Define an inner product on $V$ by
   $$\langle p, q \rangle = \frac{1}{2} \int_{-1}^{1} p(x)q(x)dx.$$  
   (a) Find an orthonormal basis for $V$ consisting of polynomials of degree 0, 1 and 2, respectively.
   (b) Find the second degree polynomial that solves the minimization problem
   $$\min_{p \in V} \int_{-1}^{1} (p(x) - x^3)^2 dx.$$  

4. Two $n \times n$ real matrices $A, B$ are said to be *similar* if there is an invertible matrix $M$ such that
   $$A = MBM^{-1}.$$  
   Describe all equivalence classes of $2 \times 2$ matrices.
Part II. Advanced Calculus

1. Find the highest point of intersection of the sphere \( x^2 + y^2 + z^2 = 30 \) and the cone \( x^2 + 2y^2 - z^2 = 0 \).

2. Let \( D \subset \mathbb{R}^n \) be a closed unbounded subset, and let \( f : D \to \mathbb{R}^m \) be a continuous function. Suppose \( f(x) \to 0 \) as \( ||x|| \to \infty \) (\( x \in D \)). Prove that \( f \) is uniformly continuous on \( D \).

3. Let \( S \) be a closed, oriented surface in \( \mathbb{R}^3 \) with unit normal vector \( \vec{n} \). Let \( \vec{c} \in \mathbb{R}^3 \) be a constant vector field. Show that the surface integral
   \[
   \int_S \vec{c} \cdot \vec{n} \, dS
   \]
   is identically zero.

4. Find domain of convergence of \( \sum_{n=1}^{\infty} \frac{(n + x)^n}{n^{n+x}} \).

5. Let \( f \) be a strictly increasing, continuous function on the closed interval \([a, b]\). Then there is a uniquely defined inverse function \( g \) defined on the interval \([f(a), f(b)]\), i.e. \( g(f(x)) = x \) for all \( x \in [a, b] \) (you do not have to prove this).
   Prove that \( g \) is continuous.