Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

1. Prove or Disprove:
   (a) The direct sum of two non-orientable real line bundles over a real manifold is orientable.
   (b) An orientable rank 2 real vector bundle over a real manifold is trivializable.
   (c) The tangent bundle of an orientable 2-dimensional real manifold can be given the structure of a complex line bundle.

2. Let $M$ be a real manifold with vanishing first deRham cohomology. Show that every real/complex line bundle $L \to M$ admits a flat connection. Are all such line bundles trivializable?

3. Let $(M,g)$ be a Riemannian manifold. For $f \in C^\infty(M,\mathbb{R})$ define its gradient by $g(\text{grad } f, -) = df$. In the special case when $\dim M = 3$ and $M$ oriented define the rotation of a vector field $X \in \mathfrak{X}(M)$ by $g(\text{rot } X, -) = *g(X, -)$, where $*$ is the Hodge star operator. Show that
   (a) $\text{rot grad } f = 0$.
   (b) If the first deRham cohomology of $M$ is trivial, then every rotation free vector field $X$ is a gradient field, $X = \text{grad } f$.

4. Consider the manifold $M = \mathbb{R}^2 \times S^1 \times S^1$ with “coordinates” $(x, y, \theta, \varphi)$. Let $E \subset TM$ be given by
   \[ dx - \cos \theta \, d\varphi = 0, \quad dy - \sin \theta \, d\varphi = 0 \]
   $M$ is the position space of a point on a wheel (modeled by a circle of radius 1) moving upright on the $(x,y)$ plane. $E$ describes the infinitesimal rolling without slipping conditions.
   (a) Show that $E$ is a rank 2 vector subbundle of $TM$. 

(b) $E$ is not (Frobenius) integrable, i.e., there are no 2-dimensional integral manifolds.

(c) Try to give an argument that every point in $M$ can be connected to every other point by an integral curve $\gamma$ of $E$, that is, $\gamma' \in E_{\gamma}$.

5. Let $V$ be a finite dimensional real vector space and let $\mathcal{E}$ denote the set of all positive definite inner products on $V$.

(a) Show that the Lie group $G = \text{GL}(V)$ of invertible linear endomorphisms of $V$ acts transitively from the right on $\mathcal{E}$ via

$$(E \cdot g)(v, w) := E(gv, gw), \quad g \in G, \ E \in \mathcal{E}$$

(b) Show that $\mathcal{E}$ has the structure of a homogeneous space $G/H$, and thus is a smooth manifold. Calculate $H$ and $\dim \mathcal{E}$.

(c) Calculate the tangent space to $G/H$ at the identity coset $[\text{id}_V] \in G/H$. Does $G$ act on this tangent space and if, how?

6. Consider the Riemannian metric $g = dx^2 + f(x)^2 dy^2$ on an open rectangle $I \times J \subset \mathbb{R}^2$, where $I, J \subset \mathbb{R}$ are open intervals and $f: I \to \mathbb{R}$ is a nowhere vanishing smooth function. Calculate the Levi-Civita connection, the curvature, and the geodesic equations for this metric. Find at least one (non-constant) geodesic in this case.

7. Use Mayer-Vietoris Theorem to compute the de Rham cohomology groups of the Klein bottle.

8. Let $M$ be a compact closed smooth manifold, and $\omega$ be a symplectic structure on $M$, i.e., $\omega$ is a closed 2-form which is non-degenerate, meaning that $X \mapsto i_X \omega$ is an isomorphism between $T_pM$ and $T^*_pM$ at every point $p \in M$. Show that the second de Rham cohomology group of $M$ is non-zero.