University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
For January, 2016

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

1. Prove or Disprove:

   (a) The direct sum of two non-orientable real line bundles over a real manifold is orientable.

   (b) An orientable rank 2 real vector bundle over a real manifold is trivializable.

   (c) The tangent bundle of an orientable 2-dimensional real manifold can be given the structure of a complex line bundle.

2. Let M be a real manifold with vanishing first deRham cohomology. Show that every real/complex line bundle $L \to M$ admits a flat connection. Are all such line bundles trivializable?

3. Let (M, g) be a Riemannian manifold. For $f \in C^\infty(M, \mathbb{R})$ define its gradient by $g(\operatorname{grad} f, -) = df$. In the special case when $\dim M = 3$ and M oriented define the rotation of a vector field $X \in \mathcal{X}(M)$ by $g(\operatorname{rot} X, \ ) = *g(X, -)$, where $*$ is the Hodge star operator. Show that

   (a) $\operatorname{rot} \operatorname{grad} f = 0$.

   (b) If the first deRham cohomology of M is trivial, then every rotation free vector field X is a gradient field, $X = \operatorname{grad} f$.

4. Consider the manifold $M = \mathbb{R}^2 \times S^1 \times S^1$ with "coordinates" (x, y, θ, φ). Let $E \subset TM$ be given by

$$dx - \cos\theta \, d\varphi = 0 \, , \quad dy - \sin\theta \, d\varphi = 0$$

   M is the position space of a point on a wheel (modeled by a circle of radius 1) moving upright on the (x, y) plane. E describes the infinitesimal rolling without slipping conditions.

   (a) Show that E is a rank 2 vector subbundle of TM.

(b) E is not (Frobenius) integrable, i.e., there are no 2-dimensional integral manifolds.

(c) Try to give an argument that every point in M can be connected to every other point by an integral curve γ of E, that is, $\gamma' \in E_\gamma$.

5. Let V be a finite dimensional real vector space and let $\mathcal{E}$ denote the set of all positive definite inner products on V.

   (a) Show that the Lie group $G = \mathbf{GL}(V)$ of invertible linear endomorphisms of V acts transitively from the right on $\mathcal{E}$ via

   $$(E \cdot g)(v, w) := E(gv, gw)\,, \quad g \in G\,,\ E \in \mathcal{E}$$

   (b) Show that $\mathcal{E}$ has the structure of a homogeneous space G/H, and thus is a smooth manifold. Calculate H and $\dim \mathcal{E}$.

   (c) Calculate the tangent space to G/H at the identity coset $[id_V] \in G/H$. Does G act on this tangent space and if, how?

6. Consider the Riemannian metric $g = dx^2 + f(x)^2 dy^2$ on an open rectangle $I \times J \subset \mathbb{R}^2$, where $I, J \subset \mathbb{R}$ are open intervals and $f\colon I \to \mathbb{R}$ is a nowhere vanishing smooth function. Calculate the Levi-Civita connection, the curvature, and the geodesic equations for this metric. Find at least one (non-constant) geodesic in this case.

7. Use Mayer-Vietoris Theorem to compute the de Rham cohomology groups of the Klein bottle.

8. Let M be a compact closed smooth manifold, and ω be a symplectic structure on M, i.e., ω is a closed 2-form which is non-degenerate, meaning that $X \mapsto i_X\omega$ is an isomorphism between T_pM and T_p^*M at every point $p \in M$. Show that the second de Rham cohomology group of M is non-zero.