Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 14, 2015

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let X in $\mathbb{R}^m$ be a union of convex sets X_i. Show that if $X_i \cap X_j \cap X_k \neq \emptyset$ for all triples i, j, k, then X is connected and simply-connected.

(2) Let $f\colon S^1 \to (0,1)$ be a continuous map.
  (a) Show that f cannot be surjective.
  (b) Show that there exists $x \in S^1$ such that $f(x) = f(-x)$.

(3) Let X be the space obtained from $\mathbb{R}$ by identifying all integer points $\mathbb{Z}$ to a single point. Show that with the quotient topology the space X is Hausdorff, connected, and non-compact.

(4) Let $\{f_\alpha\}_{\alpha\in A}$ be a family of continuous functions $X \to [0,1]$ with the property that for any $x \in X$ and any closed set $A \subset X$ with $x \notin A$, there exists an $\alpha \in A$ such that $f(x) = 0$ and $f(A) = \{1\}$. Show that

$$F\colon X \to [0,1]^A, \quad F(x) = (f_\alpha(x))_{\alpha\in A}$$

is an embedding, i.e. it is a homeomorphism onto its image.

(5) Let X be a topological space, and define a relation $\sim$ on X by declaring that $x \sim y$ if and only if there is a connected subset of X that contains both x and y.
  (a) Show that this relation is an equivalence relation, and that the equivalence classes are closed and connected.
  (b) Find the equivalence classes when X is the real line equipped with the standard topology.
  (c) Find the equivalence classes when X is the real line equipped with the topology with basis the collection of all semi-infinite intervals

$$I(a) = \{x \mid a < x\}.$$

(6) (a) Define the one-point compactification of a space.
  (b) Let X be a connected locally compact space. Prove that X is not homeomorphic to its one-point compactification.

(7) Let X be the set $\mathbb{Z}$ of integers endowed with a topology in which a subset $U \subseteq X$ is open if and only if either $X - U$ is finite or $0 \notin U$. (X is called the *countable Fort space*.)
  (a) Check that this really defines a topology on X.
  (b) Prove that X is compact.
  (c) Is X connected? Prove your answer.