Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

**Passing standard:** For Master’s level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Let $X$ in $\mathbb{R}^m$ be a union of convex sets $X_i$. Show that if $X_i \cap X_j \cap X_k \neq \emptyset$ for all triples $i, j, k$, then $X$ is connected and simply-connected.

(2) Let $f : S^1 \to (0, 1)$ be a continuous map.
   (a) Show that $f$ cannot be surjective.
   (b) Show that there exists $x \in S^1$ such that $f(x) = f(-x)$.

(3) Let $X$ be the space obtained from $\mathbb{R}$ by identifying all integer points $\mathbb{Z}$ to a single point. Show that with the quotient topology the space $X$ is Hausdorff, connected, and non-compact.

(4) Let $\{f_\alpha\}_{\alpha \in A}$ be a family of continuous functions $X \to [0, 1]$ with the property that for any $x \in X$ and any closed set $A \subseteq X$ with $x \notin A$, there exists an $\alpha \in A$ such that $f(x) = 0$ and $f(A) = \{1\}$. Show that
   
   $F : X \to [0, 1]^A, \quad F(x) = (f_\alpha(x))_{\alpha \in A}$

   is an embedding, i.e. it is a homeomorphism onto its image.

(5) Let $X$ be a topological space, and define a relation $\sim$ on $X$ by declaring that $x \sim y$ if and only if there is a connected subset of $X$ that contains both $x$ and $y$.
   (a) Show that this relation is an equivalence relation, and that the equivalence classes are closed and connected.
   (b) Find the equivalence classes when $X$ is the real line equipped with the standard topology.
   (c) Find the equivalence classes when $X$ is the real line equipped with the topology with basis the collection of all semi-infinite intervals $I(a) = \{ x \mid a < x \}$.

(6) (a) Define the one-point compactification of a space.
   (b) Let $X$ be a connected locally compact space. Prove that $X$ is not homeomorphic to its one-point compactification.

(7) Let $X$ be the set $\mathbb{Z}$ of integers endowed with a topology in which a subset $U \subseteq X$ is open if and only if either $X - U$ is finite or $0 \notin U$. ($X$ is called the countable Fort space.)
   (a) Check that this really defines a topology on $X$.
   (b) Prove that $X$ is compact.
   (c) Is $X$ connected? Prove your answer.