Complete 7 of the following 9 problems. Please show your work. The passing standards are:

- Master’s level: 60% with three questions essentially correct (including one from each part);
- Ph.D. level: 75% with two questions from each part essentially correct.

Linear Algebra

(1) The only eigenvalues of the matrix

\[
A = \begin{pmatrix}
5 & 0 & 5 & 0 \\
0 & 5 & 0 & 5 \\
0 & 5 & 0 & 5 \\
5 & 0 & 5 & 0
\end{pmatrix}
\]

are 0 and 10. Find, with a minimum of computation, the rank, nullity, trace, determinant, and bases of the eigenspaces of \(A\). Is \(A\) diagonalizable?

(2) Let \(A, B\) be \(n \times n\) matrices over a field \(k\).
   (a) Show that \(\text{rank } AB \leq \min(\text{rank } A, \text{rank } B)\).
   (b) Show that if \(\text{rank } AC = \text{rank } C\) for all \(n \times n\) matrices \(C\), then \(A\) is invertible.

(3) Let \(A\) be a (real) orthogonal \(n \times n\) matrix, so \(A^t A = I\) (where \(A^t\) denotes the transpose matrix).
   (a) Prove that \(\det A = \pm 1\).
   (b) Prove that \(x\) and \(Ax\) have the same length, for all \(x \in \mathbb{R}^n\).
   (c) Prove that all eigenvalues \(\lambda\) of \(A\) satisfy \(|\lambda| = 1\).
   (d) If \(n = 3\) and \(\det A = 1\), prove that 1 is an eigenvalue of \(A\).

(4) (a) If \(V \subset W\) are subspaces of \(\mathbb{R}^n\), and \(\dim W = \dim V + 1\), show that \(V^\perp \cap W\) is one-dimensional, where \(V^\perp\) is defined with respect to the usual inner product on \(\mathbb{R}^n\).
   (b) Given subspaces

\[
V_1 \subset V_2 \subset \cdots \subset V_{n-1} \subset \mathbb{R}^n
\]

with \(\dim V_i = i\) for all \(i\), show that there exists an orthogonal matrix \(A\) such that the span of the first \(i\) columns is \(V_i\). How many such matrices are there?
Advanced Calculus

(5) Find the average value of the function \( F(x) = \int_x^1 \sin(t^2) \, dt \) on \([0, 1]\).

(6) (a) Let \( \mathbf{F} \) be a continuously differentiable vector field and \( g \) a continuously
differentiable function on a domain in \( \mathbb{R}^3 \). Prove the following relation
involving the divergence and the gradient:
\[
\text{div}(g \mathbf{F}) = \langle \text{grad}(g), \mathbf{F} \rangle + g \text{div}(\mathbf{F}).
\]
(b) Show that there is exactly one value of \( n \) so that
\[
\mathbf{F}(x) = \frac{x}{||x||^n},
\]
which is defined everywhere on \( \mathbb{R}^3 \) except at the origin, is divergence-
free.
(c) For the \( \mathbf{F} \) you found in (b) above, compute the flux integral
\[
\int_S \mathbf{F} \cdot d\mathbf{A},
\]
where \( S \) is the sphere \((x-1)^2 + y^2 + z^2 = 100\). (Hint: use the divergence
theorem, a.k.a. Gauss’s theorem.)

(7) (a) Determine the radius of convergence \( R \) and the interval of convergence
for the power series
\[
\sum_{n=1}^\infty \frac{(-3)^n}{n} x^n.
\]
(b) If a power series \( \sum a_n x^n \) has radius of convergence \( R > 0 \) and if
\( 0 < R' < R \), prove that the series converges uniformly on \([-R', R']\).
Conclude that the series defines a continuous function on \((-R', R')\).

(8) Show that the improper Riemann integral \( \int_\pi^\infty \frac{\sin x}{x} \, dx \) converges.

(9) Let \( a_{ij}, 1 \leq i, j \leq \infty \) be real numbers.
(a) Show that if all \( a_{ij} \geq 0 \), then
\[
\sum_{i=1}^\infty \sum_{j=1}^\infty a_{ij} = \sum_{j=1}^\infty \sum_{i=1}^\infty a_{ij}.
\]
In particular, this means that the left sum converges if and only if the
right one does.
(b) Give an example to show that this need not be true if the assumption
\( a_{ij} \geq 0 \) is dropped.