(1) Let $X$ and $Y$ be spaces, with $Y$ Hausdorff.
(a) Show that for any continuous function $f : X \to Y$, the graph
\[ \Gamma_f = \{(x, f(x)) \mid x \in X\} \]
is closed in $X \times Y$.
(b) Show that if $f, g : X \to Y$ are continuous and $f|_A = g|_A$ for a dense subset $A \subset X$, then $f = g$.

(2) Let $D$ be the closed disk in $\mathbb{R}^2$, endowed with the topology generated by a basis consisting of the usual open sets, together with all sets of the form
\[ \{p\} \cup B_{1-\epsilon}(ep), \]
where $p \in S^1 = \partial D$ and $0 < \epsilon < 1$.
(a) Describe the topology $S^1$ inherits as a subspace of $D$.
(b) Show that $D$ is not compact.
(c) Show that $D$ is connected.

(3) Let $(X, d)$ be a bounded metric space. Define a metric on the countable product $\prod_{n=1}^{\infty} X$ by
\[ D((a_n), (b_n)) = \sup_n \frac{d(a_n, b_n)}{n}. \]
Prove directly from the definitions that the topology induced by $D$ is the same as the product topology.

(4) Let $X \subset \mathbb{R}^2$ be the union of all the line segments joining $(0, 0)$ to $(1/n, 1/n^2)$ for $n = 1, 2, \ldots$.
(a) Show that $X$ is homeomorphic to the one-point compactification of $(0, 1] \times \mathbb{Z}$.
(b) Show that $X$ is not homeomorphic to the quotient of $[0, 1] \times \mathbb{Z}$ identifying $\{0\} \times \mathbb{Z}$ to a point.

(5) (a) State the Lebesgue number lemma for compact metric spaces.
(b) Let $f : X \to \mathbb{R}$ be a continuous function on a compact metric space.
Prove that $f$ is uniformly continuous.

(two problems on back)
(6) Let $C$ be the set of all continuous functions $[0, 1] \to \mathbb{R}$, with the sup metric:

$$d(f, g) = \sup_{t \in [0, 1]} |f(t) - g(t)|,$$

and let $K \subset [0, 1]$ be compact and $U \subset \mathbb{R}$ be open. Show that the set

$$\{ f \in C \mid f(K) \subset U \}$$

is open in $C$.

(7) Recall that the real projective plane $\mathbb{RP}^2$ is the quotient of $S^2$ by the equivalence relation generated by $p \sim -p$ for all points $p \in S^2$.

Let $f: \mathbb{RP}^2 \to S^1$ be a continuous map. Show that there does not exist a continuous map $g: S^1 \to \mathbb{RP}^2$ so that $f \circ g$ is the identity.