Do all five problems. Sixty points are needed to pass at the Master’s level and seventy-five at the Ph.D. level.

1. (10 points) Let \( \overline{X}_1 \) and \( \overline{X}_2 \) be sample means based on two independent samples of sizes \( n_1 \) and \( n_2 \), taken from two populations with common unknown mean \( \mu \), and with known variances \( \sigma_1^2 \) and \( \sigma_2^2 \), respectively.

   (a) Find the minimum-variance unbiased estimator of \( \mu \) in the family of estimators that are linear combinations of \( \overline{X}_1 \) and \( \overline{X}_2 \). Be sure to show how you derive this estimator.

   (b) Find the variance of the estimator in (a).

2. (10 points) Let \( \hat{p}_n \) be the MLE of the MLE of the \( p \) based on \( n \) i.i.d. Bernoulli variables with probability \( p \) of success. We are particularly interested in the logistic transformation

   \[
   \theta = \log \left( \frac{p}{1-p} \right), \quad \text{and} \quad \hat{\theta}_n = \log \left( \frac{\hat{p}_n}{1 - \hat{p}_n} \right)
   \]

   (a) In terms of \( n \) and \( p \), state with brief explanation the approximate distribution of \( \hat{\theta}_n \).

   (b) Obtain an approximate 95% confidence interval for \( \theta \) for a fixed large \( n \).

3. (50 points) Let \( X_1, \ldots, X_n \) be i.i.d. each distributed Poisson(\( \lambda \)); that is \( P(X_i = x) = \lambda^x e^{-\lambda} / x! \) for \( x = 0, 1, 2, \ldots \), and \( \lambda > 0 \). Recall that \( E(X_i) = \lambda, V(X_i) = \lambda \).

   (a) Write down the likelihood function for \( \lambda \) and find the maximum likelihood estimator. Justify that your solution maximizes the likelihood function.

   (b) Find \( I_1(\lambda) \), the information for \( \lambda \) in each \( X_i \) and then the total information \( I(\lambda) \).

   (c) Find the Cramer-Rao lower bound for unbiased estimators of \( \lambda \). Using this result, will you conclude or deny that the MLE is a UMVUE for \( \lambda \)? Explain why.

   (d) Give the asymptotic distribution of the MLE \( \hat{\lambda} \) (properly centered and scaled).

   (e) Use the previous part and whatever other justification is needed to develop an approximate confidence interval for \( \lambda \).
(f) Let $\theta = \Pr\{X > 0\}$, where $X$ has the Poisson distribution as stated at the beginning of the problem.

i. Find the MLE of $\theta$, and show that (with explanations) it is consistent.
ii. Is the MLE you found in previous question unbiased? Explain your answer.

(g) Consider making inferences for $\lambda$ within the Bayesian framework. Suppose $\lambda$ has a prior distribution which is gamma with parameters $\alpha$ and $\beta$ [i.e., $\pi(\lambda) = \lambda^{\alpha-1}e^{-\lambda/\beta}/(\beta^\alpha \Gamma(\alpha))]$. Note that $E(\lambda) = \alpha \beta$ and $\text{Var}(\lambda) = \alpha \beta^2$.

i. Find the posterior distribution of $\lambda$ (It should be represented in terms of a known distribution).
ii. Find the posterior mean and posterior variance of $\lambda$
iii. Describe how to construct a 95% equal-tail posterior interval for $\lambda$.

4. (15 points)

(a) State carefully the Neyman-Pearson lemma for testing a simple null hypothesis against a simple alternative hypothesis.

(b) Based on one observation $X$ from a distribution with p.d.f. $f(x)$, derive the most powerful size 0.05 test for testing

$H_0 : f(x) = 2x, \text{ for } 0 < x < 1$

against

$H_1 : f(x) = 1, \text{ for } 0 < x < 1$.

Be sure to give the critical region explicitly.

(c) Compute the power of the test.

5. (15 points) Let $X_1, \cdots, X_n$ be a random sample from a normal distribution with mean $\mu$ and standard deviation $\sigma$, where both $\mu$ and $\sigma$ are unknown. Consider testing

$H_0 : \mu = \mu_0$ against $H_1 : \mu \neq \mu_0$,

with $\mu_0$ being a given number.

(a) Derive (step by step) the size $\alpha$ likelihood ratio test with specification of the critical value and critical region for the test in term of a well-known distribution.

(b) Explain how to construct 95% confidence interval for $\mu$ using the distribution relating to part (a).