DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
January 2014

Work all problems. Show all work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. Suppose that X is a random variable with density $\frac{3x+1}{8}$ on the interval $(0,2)$. Let Y be the area of a circle of radius X.
   (a) (8 points) Find the density of Y.
   (b) (8 points) Now suppose $X_1, \dots X_n$ are independent identically distributed random variables each with density $\frac{3x_i+1}{8}$ on the interval $(0,2)$. For large n, find the approximate distribution of $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$. Justify.
   (c) (8 points) Consider random variable $\bar{Y}^*$, which is the area of a circle of radius $\bar{X}$. For large n, find the approximate distribution of $\bar{Y}^*$. Justify.
   (d) (8 points) Consider random variables $Y_1, \dots Y_n$, where Y_i is the area of a circle with radius X_i. For large n, is the approximate distribution of $\bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i$ the same as the approximate distribution of $\bar{Y}^*$ in the previous part? Why or why not? If not, how would you find the approximate distribution of $\bar{Y}$? (you do not need to do the computations, but describe your method)

2. Suppose we toss a fair coin once and let p be the probability of heads. Let X denote the number of heads and Y denote the number of tails.
   (a) (8 points) Prove that X and Y are dependent.
   (b) (8 points) Let $N \sim Poisson(\lambda)$ and suppose we toss a fair coin N times. Let X and Y be the number of heads and tails. Show that X and Y are independent. *Hint: you may want to consider* $P(X = x|N = n)$ *and* $P(X = x, Y = y)$.

3. Suppose that (X_1, Y_1) and (X_2, Y_2) are independent, each pair distributed bivariate normal, $BVN(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix})$.
   (a) (8 points) Find the distribution of $X_1 + Y_1$ and $X_1 - Y_1$ respectively.
   (b) (8 points) Show that $Cov(X_1 + Y_1, X_1 - Y_1) = 0$.
   (c) (8 points) Let $U = (X_1+Y_1)^2+(X_2+Y_2)^2$ and $V = (X_1-Y_1)^2+(X_2-Y_2)^2$. Find the distribution of U and V respectively. *Hint: Use the fact that the sum of squares of standard normal random variables follows a Chisquare distribution.*
   (d) (8 points) As a result of (b), together with the fact that X_1+Y_1 and X_1-Y_1 are jointly distributed as BVN, they are independent. Therefore, U and V are independent. Knowing that U and V are independent, find $P(U > V)$.

The pdf of the χ^2_n distribution is

$$\frac{1}{2^{(k/2)}\Gamma(k/2)} x^{(k/2-1)} e^{-x/2}, x > 0$$

The pdf of a bivariate normal distribution is

$$f(x,y) = \frac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - 2\rho\left(\frac{x-\mu_x}{\sigma_x}\right)\left(\frac{y-\mu_y}{\sigma_y}\right) + \left(\frac{y-\mu_y}{\sigma_y}\right)^2\right]\right\}$$

4. Consider $X_1, X_2, \dots X_n \quad \sim$ iid Bernoulli(p).

   (a) (4 points) Let $\bar{X}_n = \sum_{i=1}^n X_i/n$. Find a value of τ in terms of p such that $n^{1/2}(\bar{X}_n - p)/\tau$ converges in distribution to a standard normal. Justify.

   (b) (8 points) Show that $\hat{\tau}_n = \sqrt{\bar{X}_n(1-\bar{X}_n)}$ converges in probability to τ.

   (c) (8 points) Show that $n^{1/2}(\bar{X}_n - p)/\hat{\tau}_n$ converges in distribution to a standard normal.