Complex analysis qualifying exam

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Do 8 out of the following 10 questions.

Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points with 3 questions essentially correct. To pass at the PhD level it is sufficient to have 55 points with 4 questions essentially correct.

Note: All answers should be justified carefully.

(1) (10 points).

(a) (2 points) Let $\Omega \subset \mathbb{C}$ be an open set and $a \in \Omega$ a point. Let $f\colon \Omega \setminus \{a\} \to \mathbb{C}$ be a holomorphic function. Define the residue of f at a.

(b) Find the poles and residues of the following functions.

(i) (4 points)

$$f(z) = \frac{e^z}{z(z+2)^2}.$$

(ii) (4 points)

$$g(z) = \frac{1}{\sin(z)}.$$

(2) (10 points) Consider the meromorphic function

$$f(z) = \frac{1}{z^2 - 5z + 4}.$$

Compute the Laurent series expansion

$$f(z) = \sum_{n \in \mathbb{Z}} c_n z^n$$

of f centered at $z = 0$ which is valid in a neighbourhood of $2 + i$, and determine its domain of convergence.

(3) (10 points). Let m and n be positive integers. Let γ be the closed path in the complex plane given by the circle with center $z = \frac{1}{2}$ and radius 1, traversed once counterclockwise. Compute the integral

$$\int_\gamma \frac{1}{z^m(1-z)^n} dz.$$

(4) (10 points) Compute the improper integral

$$\int_0^\infty \frac{\cos x}{x^2 + 4} dx.$$

(5) (10 points) Let $a \in \mathbb{R}$, $a > 1$. Compute the integral

$$\int_0^{2\pi} \frac{1}{a + \cos\theta} d\theta.$$

(6) (10 points) State and prove Liouville's theorem on holomorphic functions.

(7) (10 points)

(a) (6 points) Find the number of solutions (counting multiplicities) of the equation
$$z^4 - 6z + 3 = 0$$
in the annulus $A = \{z \mid 1 < |z| < 2\}$.

(b) (4 points) Show that the multiplicity of each solution in part (a) is equal to 1.

(8) (10 points) Let $\Omega = \{z \in \mathbb{C} \mid |z-1| > 2\}$ and consider the holomorphic function
$$f\colon \Omega \to \mathbb{C}, \quad f(z) = \frac{\cos(\pi z)}{z(z-2)}.$$
Show carefully that there exists a holomorphic function $g\colon \Omega \to \mathbb{C}$ such that $g' = f$.

(9) (10 points) Let $H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$ be the upper half plane. For each of the following regions Ω, determine a holomorphic bijection $f\colon \Omega \to H$.

(a) (3 points) $\{z = x + iy \in \mathbb{C} \mid 0 < x < 1\}$.

(b) (7 points) $\{z = x + iy \in \mathbb{C} \mid |z| < 1 \text{ and } x + y > 1\}$.

(10) (10 points) Let
$$D = \{z \in \mathbb{C} \mid |z| < 1\}$$
be the unit disc and let
$$H = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$
be the upper half plane. Let $f\colon D \to H$ be a holomorphic function such that $f(0) = i$. Show that
$$|f(z)| \leq \frac{1 + |z|}{1 - |z|}$$
for all $z \in D$.