Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. For the transcritical bifurcation:
   - Write the normal form.
   - Sketch the bifurcation diagram (using solid lines for stable fixed points and dashed lines for unstable ones).
   - For a field \( \dot{x} = f(x, r) \)
     provide the conditions near a critical point \((x^*, r_c)\) under which we can consider a bifurcation to be transcritical.
   - Apply these conditions to the problem (and determine its bifurcation diagram)
     \[ \dot{x} = x(r - e^x) \]

2. Consider the weakly perturbed oscillator:

   \[ \ddot{x} + x = -2\epsilon \dot{x} \]
   - Solve the system exactly with initial conditions \(x(0) = 0\) and \(\dot{x}(0) = 1\).
   - Attempt to solve it by regular perturbation theory \(x(t) = x_0(t) + \epsilon x_1(t) + \ldots\) and obtain the first two orders \((x_0(t)\) and \(x_1(t))\). Explain the problem arising. Hint: For the inhomogeneous solution in \(x_1(t)\), try \(x_1 = Ct \sin(t)\).
   - Try to solve for the leading order \(x_0\), by the same perturbative expansion, using also two-timing \(\tau = t\) and \(T = \epsilon t\). Does that give a better approximation and why?
3. (a) Consider the problem $u_{tt} = c^2 u_{xx}$ in the infinite line with the initial conditions: $u(x,0) = \phi(x) = 0$, $u_t(x,0) = \psi(x) = xe^{-x^2}$.
(a) Find the solution to this PDE at all times.
(b) Sketch the solution of the PDE at $t = 0$ and at a large time $t >> 0$.
(b) Solve the heat equation with convection:
$$u_t - ku_{xx} + Vu_x = 0$$
for $-\infty < x < \infty$ and $u(x,0) = \phi(x)$.

4. Solve the 2d Laplace equation
$$u_{xx} + u_{yy} = 0$$
in the disk $r < a$ with the boundary condition
$$u = 1 + 3 \sin(\theta)$$
on the boundary of the domain at $r = a$. Show all the details of your calculation.

5. Solve the equation
$$u_t = ku_{xx}$$
with boundary condition $u(0,t) = u(L,t) = 0$ and initial condition $u(x,0) = x$. Then use Parseval’s identity for the Fourier series of $f(x) = x$ to compute $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

6. (a) Show that the eigenvalue problem
$$\begin{cases} -f''(x) = \lambda f(x) \\ f'(0) = a_0 f(0), \quad f'(L) = -a_L f(L), \end{cases}$$
cannot have negative eigenvalues when $a_0 > 0$ and $a_L > 0$.
(b) Again, show that $\lambda$ cannot be negative for
$$\begin{cases} f'''(x) = \lambda f(x) \\ f(0) = f(L) = f''(0) = f''(L) = 0 \end{cases}.$$
7. Consider the rabbit-sheep problem for \( x > 0 \) and \( y > 0 \):

\[
\begin{align*}
\dot{x} &= x(3 - x - y) \\
\dot{y} &= y(2 - x - y)
\end{align*}
\]

(a) Find the fixed points.
(b) Classify their stability and sketch the phase plane.
(c) Explain why there can not be any limit cycles in this system.