Do 7 of the following 9 problems. **Show your work!**

Passing Standard:
- Master’s level: 60% with three questions essentially complete (including at least one from each part)
- Ph. D. level: 75% with two questions from each part essentially complete

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**Part I. Linear Algebra**

1. Let $a,b$ be real numbers. Determine the rank of the matrix \[
\begin{pmatrix}
1 & 1 & 1 \\
1 & a & b \\
1 & a^2 & b^2
\end{pmatrix}.
\]

**Note:** The answer will depend on the values of $a$ and $b$; make sure you address all possible cases.

2. Denote by $M_n(\mathbb{C})$ the set of all $n \times n$ complex matrices, and by $\mathbb{C}^n$ the $n$-dimensional complex vector space. For any $A \in M_n(\mathbb{C})$, define

\[ ||A|| := \sup_{\|\vec{x}\| \neq 0} \frac{|A\vec{x}|}{|\vec{x}|}, \quad \rho(A) := \max\{|\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } A\}. \]

(a) Exactly one of the following statements is true:
- $\rho(A) \leq ||A||$ for all $A \in M_n(\mathbb{C})$;
- $\rho(A) \geq ||A||$ for all $A \in M_n(\mathbb{C})$.

Determine which statement is true and give a proof (you do not need to give a counter-example for the other statement).

(b) Determine all $A \in M_n(\mathbb{C})$ for which $\rho(A) = ||A||$. **Justify your reasoning.**

3. Let

\[ A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}. \]

Show that every real matrix $B$ such that $AB = BA$ has the form $sI + tA$ where $s,t \in \mathbb{R}$ and $I$ is the identity matrix.

4. A square matrix $A$ with complex entries is called skew-Hermitian if $A^T = -\bar{A}$. Here, $A^T$ is the transpose of $A$ and $\bar{A}$ is the complex conjugate of $A$.

(a) Suppose $A$ and $B$ are $n \times n$ skew-Hermitian matrices, and $a,b$ are complex numbers. Under what conditions is $C = aA + bB$ a skew-Hermitian matrix?

(b) Show that every eigenvalue of a skew-Hermitian matrix is purely imaginary, i.e. the real part is zero. **Justify your reasoning.**
Part II. Advanced Calculus

1. Let $f(x)$ be a differentiable real valued functions on $\mathbb{R}$. Show that all tangent planes to the surface defined by
   $$z = xf(y/x) \quad (\text{for } x \neq 0)$$
intersects at a common point (this common point could have $x = 0$).

2. Let $W$ be the three-dimensional region under the graph of $f(x, y) = e^{x^2+y^2}$ and above the annular region in the $xy$-plane defined by $1 \leq x^2 + y^2 \leq 2$.
   (a) Find the volume of $W$.
   (b) Find the flux of the vector field $\mathbf{F}(x, y, z) = (2x - xy)\mathbf{i} - y\mathbf{j} + yz\mathbf{k}$ out of the region $W$.

3. For $x > 0$, define
   $$f(x) = \int_0^\infty t^{x-1}e^{-t}dt = \int_0^\infty \frac{t^xe^{-t}}{t} dt.$$
   (a) Show that $f$ is continuous.
   (b) Use integration by part to show that $f(x + 1) = xf(x)$.
   **Note:** The function $f$ is defined by an improper integral; make sure you address all convergence issues in your argument.

4. For each positive integer $n$, define
   $$a_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - \ln(n).$$
   (a) Show that $a_n > a_{n+1}$ for every $n \geq 1$.
   (b) Define $b_n = a_n - (1/n)$. Show that $b_{n+1} > b_n$ for every $n \geq 1$.
   (c) Show that the two sequences $\{a_n\}_n$ and $\{b_n\}_n$ are both convergent and that they both converge to the same finite value.

5. Let $\vec{a}, \vec{b}, \vec{c}$ be three distinct points in $\mathbb{R}^n$. Define a function $\mathbb{R}^n \to \mathbb{R}$ by
   $$f(\vec{x}) = |\vec{x} - \vec{a}|^2 + |\vec{x} - \vec{b}|^2 + |\vec{x} - \vec{c}|^2.$$ 
Find all point(s) $\vec{x} \in \mathbb{R}^n$ at which $f$ reaches its minimum and find the minimum value. **Justify your reasoning.**