University of Massachusetts
Department of Mathematics and Statistics
Basic Exam: Topology
January 18, 2013

Answer 5 out of the following 7 problems. Indicate clearly which questions you want graded. Justify all your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

Problem 1. Let $\mathbb{R}_\ell$ denote the reals with the lower-limit topology and $\mathbb{R}_\ell^2$ the plane with the product topology $\mathbb{R}_\ell \times \mathbb{R}_\ell$. (Recall that a basis for the lower limit topology are the intervals $[a, b)$ and (a, b) for reals $a < b$.)

a) Find the closure of $(0, 1)$ in $\mathbb{R}_\ell$.
b) Prove that $\mathbb{R}_\ell$ is not locally compact.
c) Let $A = \{(x, -x) : x \in \mathbb{R}\}$. Describe the subspace topology induced by $\mathbb{R}_\ell^2$ on A.

Problem 2. Suppose that $K_1 \supset K_2 \supset \cdots$ is a sequence of nonempty, compact, connected subsets of $\mathbb{R}^m$, $m \geq 1$. Show that the intersection

$$K = \bigcap_{n=1}^{\infty} K_n$$

is also nonempty, compact, and connected.

Problem 3. Prove that a countable product of sequentially compact spaces is sequentially compact. (If you use Tychonoff's Theorem you must prove it...)

Problem 4. Give examples of the following.

a) A complete and bounded metric space which is not compact.
b) A compact subset of a topological space X which is not closed.

Problem 5. Prove the following statements.

a) The space $X = \{f : [0, 1] \to [0, 1]\}$ with the topology of pointwise convergence (also known as point-open topology) is not metrizable.
b) If $f, g \colon \mathbb{RP}^2 \to S^1 \times S^1$ are continuous then f is homotopic to g.

Problem 6. Let $p\colon \tilde{X} \to X$ be a covering map. Let Y be connected and $y_0 \in Y$. Let $f, g\colon Y \to \tilde{X}$ be continous maps such that:

- $f(y_0) = g(y_0)$, and
- $p \circ f = p \circ g$.

Prove that $f = g$.

Problem 7. Let $D^* = \{(x, y) \in \mathbb{R}^2 : 0 < x^2 + y^2 \leq 1\}$. Let S^1 be the circle and

$$X = D^* \cup_f S^1$$

where the attaching map is given by $f(\cos t, \sin t) = (\cos 5t, \sin 5t)$. Compute $\pi_1(X)$. Give your answer in terms of generators and relations. Is $\pi_1(X)$ an abelian group?