Department of Mathematics and Statistics
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Basic Exam - Complex Analysis

January 2013

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$. Show using contour integration that

$$\int_0^\infty \frac{x^\alpha}{(1+x^2)} dx = \frac{\pi}{2\cos(\pi\alpha/2)}.$$

2. Let $g\colon \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $g(i) \neq g(-i)$. Define domains

$$\Omega_1 = \{z \in \mathbb{C} \mid |z| < 1\}$$

and

$$\Omega_2 = \{z \in \mathbb{C} \mid |z| > 1\}.$$

   (a) Does there exist a holomorphic function $f\colon \Omega_1 \to \mathbb{C}$ such that $f'(z) = \frac{g(z)}{z^2+1}$?

   (b) Does there exist a holomorphic function $f\colon \Omega_2 \to \mathbb{C}$ such that $f'(z) = \frac{g(z)}{z^2+1}$?

   Justify your answers carefully.

3. (a) State Rouché's theorem.

   (b) Determine the number of zeroes of the function $f(z) = 2z^2 + \sin(z)$ in the open unit disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$ and show that all the zeroes are simple.

4. Define domains

$$\Omega_1 = \{z \in \mathbb{C} \mid -\pi/2 < Im(z) < \pi/2\},$$
$$\Omega_2 = \{z \in \mathbb{C} \mid |z-i| > 1 \text{ and } |z-2i| < 2\},$$

and

$$\mathbb{H} = \{z \in \mathbb{C} \mid Im(z) > 0\}.$$

   (a) Describe a one-to-one and onto holomorphic map $f_1\colon \Omega_1 \to \mathbb{H}$.

   (b) Using part (a) or otherwise, describe a one-to-one and onto holomorphic map $f_2\colon \Omega_2 \to \mathbb{H}$.

   Justify your answers carefully.

5. Let γ denote the circle $\{z \in \mathbb{C} \mid |z| = 2\}$ traversed once in the counter-clockwise direction. Compute the following contour integrals.

   (a) $\displaystyle\int_\gamma \frac{1+z}{1-\cos z} dz.$

   (b) $\displaystyle\int_\gamma z e^{1/z} dz.$

6. Prove that the series $\displaystyle\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ defines a meromorphic function $f(z)$, periodic with period 1, over the complex plane.

7. Prove that every one-to-one holomorphic map f from the upper-half-plane $\mathbb{H} := \{x + iy \ : \ x, y \in \mathbb{R}, \ y > 0\}$ onto itself is a fractional linear transformation with real coefficients and positive determinant. That is, f can be written in the form:

$$f(z) \quad = \quad \frac{az+b}{cz+d},$$

   where $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$.

8. Evaluate the integral $\displaystyle\int_0^\pi \frac{\cos^2(\theta)}{2+\cos(\theta)} d\theta$. Justify all your steps.

9. Let $\operatorname{Log}(z)$ be the principal branch of the logarithm function and C the circle of radius 2 centered at the origin positively oriented. Evaluate the integral $\displaystyle\int_C z^n \operatorname{Log}\left(1 - \frac{1}{z}\right) dz$ for all integers n. Carefully justify your answer.

10. Prove or disprove.

   (a) There exists a one to one conformal map from the upper half plane

$$\mathbb{H} = \{z \ : \ Im(z) > 0\}$$

   onto the complex plane $\mathbb{C}$.

   (b) If $f(z)$ is analytic in a domain D and $u(x, y) = \operatorname{Re}(f(x + iy))$, then the function $H(x, y) = e^{u(x,y)}$ is harmonic in D.

   (c) Let f be an entire function. Then $\displaystyle\lim_{n\to\infty} \left(\frac{|f^{(n)}(0)|}{n!}\right)^{1/n} = 0.$