Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master’s level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Let $\alpha \in \mathbb{R}$, $0 < \alpha < 1$. Show using contour integration that
\[
\int_0^\infty \frac{x^\alpha}{(1 + x^2)} dx = \frac{\pi}{2 \cos(\pi \alpha/2)}.
\]

2. Let $g: \mathbb{C} \to \mathbb{C}$ be a holomorphic function such that $g(i) \neq g(-i)$. Define domains
\[\Omega_1 = \{ z \in \mathbb{C} | |z| < 1 \}\]
and
\[\Omega_2 = \{ z \in \mathbb{C} | |z| > 1 \}\].

(a) Does there exist a holomorphic function $f: \Omega_1 \to \mathbb{C}$ such that $f'(z) = \frac{g(z)}{z^2+1}$?

(b) Does there exist a holomorphic function $f: \Omega_2 \to \mathbb{C}$ such that $f'(z) = \frac{g(z)}{z^2+1}$?

Justify your answers carefully.

3. (a) State Rouché’s theorem.

(b) Determine the number of zeroes of the function $f(z) = 2z^2 + \sin(z)$ in the open unit disc $D = \{ z \in \mathbb{C} | |z| < 1 \}$ and show that all the zeroes are simple.

4. Define domains
\[\Omega_1 = \{ z \in \mathbb{C} | -\pi/2 < \text{Im}(z) < \pi/2 \}, \]
\[\Omega_2 = \{ z \in \mathbb{C} | |z-i| > 1 \text{ and } |z - 2i| < 2 \}, \]
and
\[\mathbb{H} = \{ z \in \mathbb{C} | \text{Im}(z) > 0 \}. \]

(a) Describe a one-to-one and onto holomorphic map $f_1: \Omega_1 \to \mathbb{H}$.

(b) Using part (a) or otherwise, describe a one-to-one and onto holomorphic map $f_2: \Omega_2 \to \mathbb{H}$.

Justify your answers carefully.
5. Let $\gamma$ denote the circle $\{z \in \mathbb{C} \mid |z| = 2\}$ traversed once in the counter-clockwise direction. Compute the following contour integrals.

(a) $\int_{\gamma} \frac{1 + z}{1 - \cos z} \, dz$.

(b) $\int_{\gamma} z e^{1/z} \, dz$.

6. Prove that the series $\sum_{n=-\infty}^{\infty} \frac{1}{(z-n)^2}$ defines a meromorphic function $f(z)$, periodic with period 1, over the complex plane.

7. Prove that every one-to-one holomorphic map $f$ from the upper-half-plane $\mathbb{H} := \{x + iy : x, y \in \mathbb{R}, y > 0\}$ onto itself is a fractional linear transformation with real coefficients and positive determinant. That is, $f$ can be written in the form:

$$f(z) = \frac{az + b}{cz + d},$$

where $a, b, c, d \in \mathbb{R}$, and $ad - bc = 1$.

8. Evaluate the integral $\int_{0}^{\pi} \frac{\cos^2(\theta)}{2 + \cos(\theta)} \, d\theta$. Justify all your steps.

9. Let $\text{Log}(z)$ be the principal branch of the logarithm function and $C$ the circle of radius 2 centered at the origin positively oriented. Evaluate the integral $\int_{C} z^n \text{Log} \left( 1 - \frac{1}{z} \right) \, dz$ for all integers $n$. Carefully justify your answer.

10. Prove or disprove.

(a) There exists a one to one conformal map from the upper half plane

$$\mathbb{H} = \{z : \text{Im}(z) > 0\}$$

onto the complex plane $\mathbb{C}$.

(b) If $f(z)$ is analytic in a domain $D$ and $u(x, y) = \text{Re}(f(x + iy))$, then the function $H(x, y) = e^{u(x,y)}$ is harmonic in $D$.

(c) Let $f$ be an entire function. Then $\lim_{n \to \infty} \left( \frac{|f^{(n)}(0)|}{n!} \right)^{1/n} = 0$. 

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