DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
MASTER'S OPTION EXAM — APPLIED MATH
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Do 5 of the following questions. Each question carries the same weight. Passing level is 60% and at least two questions substantially correct.

1) Consider the dynamical system as a function of its parameter μ:

$$\dot{x} = \mu x + y + \sin(x)$$

$$\dot{y} = x - y$$

1. Illustrate that the system has a bifurcation at $\mu = -2$ and identify its nature.
2. Does the system have any additional bifurcations ? Can you sketch a full bifurcation diagram of x as a function of μ ?

2) Consider the rabbit-sheep problem for $x > 0$ and $y > 0$:

$$\dot{x} = x(5 - x - 2y)$$

$$\dot{y} = y(4 - x - y)$$

1. Find the fixed points.
2. Classify their stability and sketch the phase plane.
3. Explain why there can not be any limit cycles in this system.

3) Consider the problem $u_t = u_{xx}$ with $u_x(0,t) = u_x(\pi,t) = 0$. Solve the PDE (by separating the variables and applying the boundary conditions) to obtain the most general possible solution satisfying these boundary conditions. Then, find the unique solution that satisfies

$$u(x,0) = 5\cos(2x) - 3\cos(7x)$$

4a) Prove the uniqueness of the solution for the problem $u_{tt} = c^2 u_{xx}$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$, $u_x(0,t) = 0$, $u_x(l,t) = 0$ in (0,l), by means of the energy method.

Notice: you should *prove* that the energy is conserved.

4b) Prove the uniqueness of the solution for the problem $u_t = k u_{xx}$, $u(x,0) = \phi(x)$, $u(0,t) = u(l,t) = 0$ in two ways:

1. using the maximum principle;
2. using the energy method.

5. The viscous Burgers' equation for $u(x,t)$,

$$u_t + \left(\frac{1}{2}\,u^2\right)_x = \epsilon\, u_{xx}, \qquad (\, x \in R^1,\ t > 0\,)$$

is a fundamental equation for nonlinear viscous flows.

(a) Make the substitution

$$w(x,t) = \int_{-\infty}^{x} u(\xi,t)\, d\xi,$$

and derive the PDE satisfied by $w(x,t)$.

(b) Now make the substitution

$$w(x,t) = \alpha \log \phi(x,t)\,, \qquad \text{where } \alpha \text{ is a positive constant,}$$

and derive an equivalent PDE for ϕ.

(c) Conclude that for an appropriate choice of constant α, solutions u of Burgers' equation are in 1-1 correspondence with solutions ϕ of the heat equation. This is known as the "Cole-Hopf transformation."

6. Consider the PDE for $u = u(x,t)$,

$$u_t + [(1-u)u]_x = 0, \qquad (\, x \in R^1,\ t > 0\,)\,.$$

(a) Using the method of characteristics, solve the equation for the initial data

$$u_1(x,0) = \begin{cases} 0 & \text{if} \quad x \le 0 \\ x & \text{if} \quad 0 < x \le 1 \\ 1 & \text{if} \quad x > 0 \end{cases}$$

and

$$u_2(x,0) = \begin{cases} 1 & \text{if} \quad x \le 0 \\ 1-x & \text{if} \quad 0 < x \le 1 \\ 0 & \text{if} \quad x > 0 \end{cases}$$

(b) Do solutions exist globally in time in both cases? Explain your answer and plot solutions for suitably selected typical times.

7. A nonlinear oscillator with displacement $x \in R^1$ is governed by the ODE:

$$\frac{d^2x}{dt^2} + \frac{dV}{dx} = 0\,, \qquad \text{with potential} \quad V = \frac{1}{2}x^2 - \frac{1}{3}x^3\,.$$

(a) Reformulate this dynamical equation as a two-dimensional system of first-order equations. Determine the equilibrium points of the system.

(b) Analyze the stability of each equilibrium point and sketch the entire phase portrait.

(c) Find a function $H = H(x, \dot{x})$ on the phase plane that is constant on each solution trajectory.