Do 5 of the following 8 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

1. Prove or disprove the following statements:
   (a) The tangent bundle $T(S^2 \times S^1)$ is trivial.
   (b) If a 1-form $\alpha$ on a smooth manifold $M$ is nowhere zero and $\theta$ is another 1-form such that $\theta \wedge \alpha = 0$, then there exists $f \in C^\infty(M)$ such that
   \[ \theta = f \alpha. \]
   (c) If a 1-form $\alpha$ on $M = \mathbb{R}^2 \setminus \{0\}$ such that $d\alpha = 0$, then there exists $f \in C^\infty(\mathbb{R}^2 \setminus \{0\})$ such that $\alpha = df$.

2. Let $M_n(\mathbb{R})$ be the space of real $n \times n$ matrices. Show that the identity matrix $I$ is a regular value of the smooth map $M_n(\mathbb{R}) \to S_n(\mathbb{R}) : A \mapsto AA^t$, where $S_n(\mathbb{R})$ is the space of symmetric $n \times n$ matrices. Use this to prove $O_n(\mathbb{R}) = \{ A : AA^t = I \} \subset M_n(\mathbb{R})$ is a smooth submanifold of dimension $n(n-1)/2$.

3. Let $X_1, X_2, X_3$ be smooth vector fields defined in a neighborhood $U$ of $0 \in \mathbb{R}^3$. Suppose
   (a) $X_1(0), X_2(0), X_3(0)$ are linearly independent;
   (b) $[X_i, X_j] \equiv 0$ for all $i, j$.
   Show that there are local coordinate functions $x_1, x_2, x_3$ defined in a neighborhood $V \subset U$ of $0 \in \mathbb{R}^3$ such that
   \[ X_i = \frac{\partial}{\partial x_i}, \quad \text{for } i = 1, 2, 3. \]

4. Consider the vector space $\mathbb{R}^3$ with the cross product $\times : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$.
   (a) Show that $(\mathbb{R}^3, \times)$ is a Lie algebra.
   (b) Determine the Lie group $G$ such that (i) $G$ is simply connected, (ii) the Lie algebra of $G$ is isomorphic to $(\mathbb{R}^3, \times)$.
5. Use the Mayer-Vietoris sequence and induction to compute the de Rham cohomology groups of the $n$-sphere $S^n$ and the real projective space $\mathbb{RP}^n$.

6. Let $M$ be a closed $2n$-dimensional smooth manifold and let $\omega$ be a closed 2-form on $M$ which is non-degenerate, i.e., for any $p \in M$, $X \in T_p M$, $X \mapsto i_X \omega(p)$ defines an isomorphism between $T_p M$ and $T^*_p M$. Show that $H^{2k}_{dR}(M) \neq 0$ for any $0 \leq k \leq n$.

7. Prove that any smooth, proper function $f : \mathbb{C} \setminus \{0, 1\} \rightarrow \mathbb{R}$ has a critical point (recall that $f$ is proper if the pre-image of any compact set is compact).

8. Endow the upper half-plane

$$H = \{ (x, y) : y > 0 \}$$

with the Riemannian metric $(dx^2 + dy^2)/y^2$.

(a) Show that vertical rays

$$\{(x, y) : x = a \} \subset H$$

and half-circles

$$\{(x, y) : (x - b)^2 + y^2 = c \} \subset H$$

are geodesics.

(b) Compute the Gauss curvature of $H$.

(c) Apply the Gauss-Bonnet formula to find the area of the noncompact domain

$$\{(x, y) : -1 < x < 1, x^2 + y^2 > 1 \} \subset H.$$