DEPARTMENT OF MATHEMATICS AND STATISTICS
UMASS - AMHERST
BASIC EXAM - PROBABILITY
WINTER 2012

Work all problems. Show your work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters level and 75 to pass at the Ph.D. level

1. (20 points) There is more information in the joint distribution of two random variables than can be discerned by looking only at their marginal distributions. Consider two random variables, X_1 and X_2, each distributed binomial$(1, \pi)$, where $0 < \pi < 1$. Let $Q_{ab} = P\{X_1 = a, X_2 = b\}$.

   (a) In general, show that $0 \leq Q_{11} \leq \pi$. In particular, evaluate Q_{11} in three cases: where X_1 and X_2 are independent, where $X_2 = X_1$, and where $X_2 = 1 - X_1$.

   (b) For each case in (a), evaluate Q_{00}.

   (c) If $P\{X_2 = 1|X_1 = 0\} = \alpha$ and $P\{X_2 = 0|X_1 = 1\} = \beta$, then express π, Q_{00}, and Q_{11} in terms of α and β.

   (d) In part (c), find the correlation between X_1 and X_2 in terms of α and β.

2. (20 points) Let Y_1 and Y_2 have the joint probability density function:

$$\begin{aligned} f(y_1, y_2) &= k(1 - y_2), 0 \leq y_1 \leq y_2 \leq 1 \\ &= 0, \text{ otherwise.} \end{aligned}$$

   (a) Find k.

   (b) Find the marginal density functions for Y_1 and Y_2.

   (c) Are Y_1 and Y_2 independent? Why or why not?

   (d) Find the conditional density function of Y_2 given $Y_1 = y_1$.

   (e) Find $Pr(Y_2 \geq 3/4|Y_1 = 1/2)$.

3. (20 points) Suppose that the random variable Y has a Poisson distribution with mean λ. The probability mass function is

$$f(y|\lambda) = \frac{e^{-\lambda}\lambda^y}{y!}, \text{ for } \lambda > 0, y = 0, 1, 2, \ldots$$

   (a) Prove that $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}$.

   (b) Find the moment generating function of Y.

   (c) Suppose that Y_1 and Y_2 are independent Poisson random variables with means λ_1 and λ_2 respectively.

i. Derive the distribution of $Z = Y_1 + Y_2$.

ii. Derive the distribution of $Y_1|Z = k$.

4. (20 points) Suppose $X_i, i = 1, \ldots, n$ are independent and each has mean μ and variance $\sigma^2 < \infty$. Let $Z_i = X_i - \mu$.

   (a) Let $S_n = Z_1 + \ldots + Z_n$. Prove that $\lim_{n\to\infty} Pr(|S_n/n| > 0) = 0$.

   (b) Find $f(n, \sigma)$, a function of n and σ, so that $Z_n = f(n, \sigma)S_n$ converges in distribution to a standard normal distribution as $n \to \infty$.

   (c) Approximately what is $\lim_{n\to\infty} Pr(|Z_n| > 1.645)$?

5. (20 points) Suppose we flip coins. Let the random variable $X_i = 1$ if the ith flip is a head and 0 otherwise. Assume that the X_is are independent Bernoulli random variables with $Pr(X_i = 1) = \pi$. Let N be the number of flips required to get the first head ($N = 1, 2, \ldots$).

   (a) What is $E(N|X_1 = i), i = 0, 1$?

   (b) Use the result from part (a) and the law of iterated expectations to derive $E(N)$.

   (c) What is the the probability mass function of N?

   (d) Let $M = N - k$, where $k > 0$ is a constant integer. Derive $Pr(M > m|N > k)$.

   (e) What is the probability mass function of M?