1. (20 points) There is more information in the joint distribution of two random variables than can be discerned by looking only at their marginal distributions. Consider two random variables, \( X_1 \) and \( X_2 \), each distributed binomial\((1, \pi)\), where \( 0 < \pi < 1 \). Let \( Q_{ab} = P\{X_1 = a, X_2 = b\} \).

(a) In general, show that \( 0 \leq Q_{11} \leq \pi \). In particular, evaluate \( Q_{11} \) in three cases: where \( X_1 \) and \( X_2 \) are independent, where \( X_2 = X_1 \), and where \( X_2 = 1 - X_1 \).

(b) For each case in (a), evaluate \( Q_{00} \).

(c) If \( P\{X_2 = 1|X_1 = 0\} = \alpha \) and \( P\{X_2 = 0|X_1 = 1\} = \beta \), then express \( \pi \), \( Q_{00} \), and \( Q_{11} \) in terms of \( \alpha \) and \( \beta \).

(d) In part (c), find the correlation between \( X_1 \) and \( X_2 \) in terms of \( \alpha \) and \( \beta \).

2. (20 points) Let \( Y_1 \) and \( Y_2 \) have the joint probability density function:

\[
f(y_1, y_2) = \begin{cases} 
  k(1 - y_2), & 0 \leq y_1 \leq y_2 \leq 1 \\
  0, & \text{otherwise}.
\end{cases}
\]

(a) Find \( k \).

(b) Find the marginal density functions for \( Y_1 \) and \( Y_2 \).

(c) Are \( Y_1 \) and \( Y_2 \) independent? Why or why not?

(d) Find the conditional density function of \( Y_2 \) given \( Y_1 = y_1 \).

(e) Find \( Pr(Y_2 \geq 3/4|Y_1 = 1/2) \).

3. (20 points) Suppose that the random variable \( Y \) has a Poisson distribution with mean \( \lambda \). The probability mass function is

\[
f(y|\lambda) = \frac{e^{-\lambda} \lambda^y}{y!}, \text{ for } \lambda > 0, y = 0, 1, 2, \ldots
\]

(a) Prove that \( e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} \).

(b) Find the moment generating function of \( Y \).

(c) Suppose that \( Y_1 \) and \( Y_2 \) are independent Poisson random variables with means \( \lambda_1 \) and \( \lambda_2 \) respectively.
i. Derive the distribution of $Z = Y_1 + Y_2$.
ii. Derive the distribution of $Y_1 | Z = k$.

4. (20 points) Suppose $X_i, i = 1, \ldots, n$ are independent and each has mean $\mu$ and variance $\sigma^2 < \infty$. Let $Z_i = X_i - \mu$.
   
   (a) Let $S_n = Z_1 + \ldots + Z_n$. Prove that $\lim_{n \to \infty} Pr(|S_n/n| > 0) = 0$.
   
   (b) Find $f(n, \sigma)$, a function of $n$ and $\sigma$, so that $Z_n = f(n, \sigma)S_n$ converges in distribution to a standard normal distribution as $n \to \infty$.
   
   (c) Approximately what is $\lim_{n \to \infty} Pr(|Z_n| > 1.645)$?

5. (20 points) Suppose we flip coins. Let the random variable $X_i = 1$ if the $i$th flip is a head and 0 otherwise. Assume that the $X_i$s are independent Bernoulli random variables with $Pr(X_i = 1) = \pi$. Let $N$ be the number of flips required to get the first head ($N = 1, 2, \ldots$).
   
   (a) What is $E(N|X_1 = i), i = 0, 1$?
   
   (b) Use the result from part (a) and the law of iterated expectations to derive $E(N)$.
   
   (c) What is the the probability mass function of $N$?
   
   (d) Let $M = N - k$, where $k > 0$ is a constant integer. Derive $Pr(M > m|N > k)$.
   
   (e) What is the probability mass function of $M$?