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Advanced Analysis Qualifying Examination
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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $\mathbb{R}$. This measure is denoted by m.
3. If a σ-algebra on $\mathbb{R}$ is not specified, use the Borel σ-algebra.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space. Let $\overline{\mathcal{M}}$ be the collection of sets of the form $E \cup Z$ where $E \in \mathcal{M}$ and $Z \subset F$ for some $F \in \mathcal{M}$ with $\mu(F) = 0$. Define $\overline{\mu}$ on $\overline{\mathcal{M}}$ by

$$\overline{\mu}(E \cup Z) = \mu(E)$$

   (a) Show that $\overline{\mathcal{M}}$ is the smallest σ-algebra which contains $\mathcal{M}$ and all subsets of elements of $\mathcal{M}$ of measure 0.

   (b) Show that $\overline{\mu}$ is measure on $\overline{\mathcal{M}}$ and this measure is complete, i.e., every subset of a set of $\overline{\mu}$ measure 0 is measurable.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function such that, for all $a \in \mathbb{R}$, we have

$$\int_{[0,a]} f dm = 0 \, .$$

Show that $f(x) = 0$ for m almost every x.

3. (a) Let X_1, X_2 be two spaces equipped with σ-algebras $\mathcal{M}_1$, $\mathcal{M}_2$ respectively. Suppose that the function $f : X_1 \to I\!R$ is $\mathcal{M}_1$-measurable. Show the function $F : X_1 \times X_2 \to I\!R$ given by

$$F(x_1, x_2) = f(x_1)\,,$$

is $\mathcal{M}_1 \times \mathcal{M}_2$ measurable where $\mathcal{M}_1 \times \mathcal{M}_2$ denotes the product σ- algebra.

(b) Let $(X, \mathcal{M}, \mu)$ be a measure space and let $f : X \to I\!R$ be a nonnegative integrable function. Show that

$$\int f \, d\mu \;=\; \mu \times m\left(\{(x,t) \in X \times I\!R, 0 \le t \le f(x)\}\right)\,. \tag{1}$$

Hint: To show that the set on the r.h.s. of (1) is measurable, use part (a).

4. Let $f : \mathbb{R} \to \mathbb{R}$ be a Lebesgue integrable function and let $g : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function. Show that

$$\lim_{t \to 0} \int f(x) \left(g(x) - g(x+t) \right) \, dm = 0 \, .$$

5. Let $\mathcal{H}$ be a real Hilbert space with inner product (x, y) and let $B : \mathcal{H} \times \mathcal{H} \to I\!R$ be a bilinear functional, i.e., $B(x, y)$ is linear in the variables x and y separately. Assume that there exist constants $K > 0$ and $d > 0$ such that B satisfies the following inequalities:

$$\begin{aligned} |B(x,y)| &\leq K||x||||y||, \quad \text{for all} \quad x, y \in \mathcal{H} && (2) \\ B(x,x) &\geq d||x||^2, \quad \text{for all} \quad x \in \mathcal{H} && (3) \end{aligned}$$

(a) Show that for each $z \in \mathcal{H}$ there is a uniquely determined $y \in \mathcal{H}$ such that $(y, x) = B(z, x)$ for all $x \in \mathcal{H}$.

(b) Prove that if the correspondence in part (a) is denoted by $y = Az$, then A is a bounded linear operator on $\mathcal{H}$ that is 1-1 and which has closed range, i.e., the subspace $\mathcal{R} = A(\mathcal{H})$ is a closed subspace of $\mathcal{H}$.

(c) Prove that $\mathcal{R} = \mathcal{H}$, and that for any bounded linear functional F on $\mathcal{H}$ there exists a unique $z \in \mathcal{H}$ such that $F(x) = B(z, x)$ for all $x \in \mathcal{H}$.

6. (a) For any given p on $1 \leq p < \infty$, find an example of an *unbounded* continuous function in $L^p(\mathbb{R})$.

(b) Show that if $1 \leq p < \infty$ and $f \in L^p(\mathbb{R})$ is a function that is uniformly continuous on $\mathbb{R}$, then $\lim_{x \to \pm\infty} f(x) = 0$.

7. Let $\mathcal{X}$ be a Banach space and let $T : \mathcal{X} \to \mathcal{X}$ be a bounded linear operator with $||T|| < 1$, where

$$||T|| = \sup_{x \neq 0} \frac{||Tx||}{||x||}.$$

Prove that the operator $I - T$ has a bounded inverse, and that

$$||(I - T)^{-1}|| \leq \frac{1}{1 - ||T||}$$

8. For a function $g : [0,1] \to I\!R$ let us by denote by $TV_0^1(g)$ the total variation of g on $0 \le x \le 1$. Suppose that $f_n(x)$ is a sequence of functions of bounded variation on $0 \le x \le 1$ such that

   (a) There exists $M < \infty$ such that $TV_0^1(f_n) \le M$ for all n.

   (b) $\lim_{n\to\infty} f_n(x) = f(x)$ pointwise on $0 \le x \le 1$.

   Prove that the limit function f has bounded variation and, in particular, that

$$TV_0^1(f) \le \liminf_n TV_0^1(f_n).$$