UNIVERSITY OF MASSACHUSETTS
Department of Mathematics and Statistics
Basic Exam - Statistics
Friday, January 14, 2011

Work all problems. 60 points are needed to pass at the Masters Level and 75 to pass at the Ph.D. level.

1. (20 PTS) Let $X_1, \ldots, X_n$ be a random sample from a uniform distribution with probability density function

$$f(x;\theta) = \frac{1}{\theta}, \quad 0 \le x \le \theta, \quad \theta > 0$$

   (a) Find the moment estimator of θ, $\tilde{\theta}$, by the method of moments.

   (b) Show that the maximum likelihood estimator for θ is $\hat{\theta} = X_{(n)} = \max\{X_1, \ldots, X_n\}$.

   (c) Compute the mean square error (MSE) for $\tilde{\theta}$ and $\hat{\theta}$, respectively [Note that the probability density function of $X_{(n)}$ is $f(x_{(n)};\theta) = \frac{n x_{(n)}^{n-1}}{\theta^n}$ where $0 \le x_{(n)} \le \theta$]

   (d) Which estimator is preferred? Justify your choice.

2. (20 PTS) Let $X_1, \ldots, X_n$ be a random sample from a Poisson distribution with probability mass function

$$f(x;\theta) = \frac{e^{-\theta}\theta^x}{x!}, \quad \theta > 0, \quad x = 0, 1, 2, \cdots.$$

   We know that the maximum likelihood estimator for θ is $\hat{\theta} = \bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

   (a) Find a complete sufficient statistic for θ.

   (b) Find the UMVUE (uniform minimum variance unbiased estimator) for θ. State clearly what general results you are applying.

   Consider the prior distribution for the parameter θ as a gamma distribution with probability density function (i.e., $\theta \sim Gamma(\alpha, \beta)$)

$$f(\theta;\alpha,\beta) = \frac{1}{\Gamma(\alpha)\beta^\alpha}\theta^{\alpha-1}e^{\theta/\beta}, \quad \theta > 0, \alpha > 0, \beta > 0$$

   Note that for $\theta \sim Gamma(\alpha, \beta)$, $E(\theta) = \alpha\beta$ and $Var(\theta) = \alpha\beta^2$.

   (c) Find the posterior distribution of θ.

   (d) Find the Bayes estimator of θ under squared error loss.

3. (15 PTS) Let $X_1, \ldots, X_n$ be a random sample from an exponential distribution with probability density function

$$f(x;\lambda) = \frac{1}{\lambda} e^{-x/\lambda}, \ 0 < x < \infty, 0 < \lambda < \infty$$

   (a) Find the maximum likelihood estimator of λ. Also show that your value is an MLE.

   (b) For a fixed large n, find a 95% confidence interval for λ using normal approximation.

   (c) For a fixed large n, describe how one can construct a 95% confidence interval for λ using the likelihood ratio statistic.

4. (20 points) Let $X_1, \ldots, X_n$ be i.i.d. Bernoulli variables with probability p of success. Let $\hat{p}_n$ be the MLE of the p. We are particularly interested in the logistic transformation

$$\theta = \log_e \left(\frac{p}{1-p} \right).$$

   (a) Find the asymptotic distribution of $\sqrt{n}(\hat{p}_n - p)$.

   (b) Find the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta)$.

   (c) In terms of n and p, state the approximate distribution of $\hat{\theta}_n$.

   (d) Obtain an approximate 95% confidence interval for θ for a fixed large n.

5. (25 points) Let $X_1, \cdots, X_{25}$ be a random sample from a normal distribution with an unknown mean μ and variance 1. Consider testing the hypotheses

$$H_0 \ : \ \mu \leq 0 \ \text{ against } \ H_1 \ : \ \mu > 0.$$

   It is known that the UMP size 0.05 test rejects H_0 iff $5\overline{X} > \ c$, for some constant c.

   (a) Find the value c.

   (b) Explain what it means to be UMP.

   (c) Construct the power function of the test, and calculate the power of the test at $\mu = 0.5$. Is the power function an increasing fuction of μ? Sketch the power function and explain it.

   (d) Should the type I error probability at $\mu = -1$ be larger or smaller than 0.05? Explain briefly.

   (e) Now suppose we set the critical value c of the test to 1.96, then what is size of the new test?