1. Let $A$ be a bounded region in $\mathbb{R}^2$ and $|A|$ be its area. The boundary of $A$ is known and, for any $(x, y) \in \mathbb{R}^2$ it is easy to determine whether $(x, y) \in A$. However, $|A|$ is not known and cannot be calculated analytically. The following method is proposed to estimate $|A|$: 

(a) Construct a rectangle $B$ that contains $A$.
(b) Generate $N$ (a large integer) points at random, uniformly, in $B$.
(c) Let $X$ be the number of generated points that lie within $A$.
(d) Use $X/N$ as an estimate of $|A|/|B|$ and $|B| \times (X/N)$ as an estimate of $|A|$.

The user can choose $B$, as long as it’s big enough to contain $A$. If the goal is to estimate $|A|$ as accurately as possible, what advice would you give the user for choosing the size of $B$? Should $|B|$ be large, small, or somewhere in between? Justify your answer. *Hint: think about Binomial distributions.*
2. You go to the bus stop to catch a bus. You know that buses arrive every 15 minutes, but you don’t know when the next is due. Let $T$ be the time elapsed, in hours, since the previous bus. Adopt the prior distribution $T \sim \text{Unif}(0, 1/4)$.

(a) Find $E[T]$.

Passengers, apart from yourself, arrive at the bus stop according to a Poisson process with rate $\lambda = 2$ people per hour; i.e., in any interval of length $\ell$, the number of arrivals has a Poisson distribution with parameter $2\ell$ and, if two intervals are disjoint, then their numbers of arrivals are independent. Let $X$ be the number of passengers, other than yourself, waiting at the bus stop when you arrive.

(b) Suppose $X = 1$. Write an intuitive argument for whether that should increase or decrease your expected value for $T$. I.e., is $E[T|X = 1]$ greater than, less than, or the same as $E[T]$?

(c) Find the density of $T$ given $X = 1$, up to a constant of proportionality. It is a truncated version of a familiar density. What is the familiar density?
3. A discrete-time Markov chain is a series of indexed random variables, 
\{X_0, X_1, X_2, \ldots\} which displays the Markov property, namely
\[
\Pr(X_{n+1} = j | X_0 = x_0, X_1 = x_1, X_2 = x_2, \ldots X_n = i) = \Pr(X_{n+1} = j | X_n = i).
\]
Consider such a Markov chain in which there are only finitely many possible \(x\)’s and in which the so-called transition probabilities are given by the matrix \(p\) such that
\[
p_{ij} = \Pr(X_{n+1} = j | X_n = i),
\]
constant for all \(n \geq 0\).

(a) Give an expression in terms of \(p\) for the probability that \(X_{n+2} = j\)
given \(X_n = i\).

(b) Give an expression in terms of \(p\) for the probability that \(X_{n+m} = j\)
given \(X_n = i\). For full credit, whenever possible, express your answer
using matrix notation rather than functions of the matrix elements.

(c) Prove that for any Markov chain, \(\Pr(X_3 = x_3 | X_0 = x_0, X_1 = x_1) = \Pr(X_3 = x_3 | X_1 = x_1)\).
4. Suppose $X_1$ and $X_2$ are random variables with joint density function $f(x_1, x_2) = c$ when $x_1 + x_2 \leq 1$ and both $x_1$ and $x_2$ are non-negative. The density $f(x_1, x_2) = 0$ otherwise. Except for part (a), purely graphical solutions will not get full credit.

(a) Draw a picture to show the $x_1$ and $x_2$ values where the density is non-zero.

(b) What is $c$?

(c) What is the probability that $X_1 > X_2$?

(d) Are $X_1$ and $X_2$ independent? Why or why not?

(e) What is the density of $Y = 1/X_1$?
5. Suppose $X_1$ and $X_2$ are independent and identically distributed random variables with density $f(x) = \lambda \exp(-\lambda x), x \geq 0,$ and $f(x) = 0$ otherwise.

(a) The moment generating function of a random variable $X$ is $M_X(t) = \mathbb{E}[e^{tX}]$. Find the moment generating function of $X_1$.

(b) Use the moment generating function to show that $Y = X_1 + X_2$ has density $f(y) = \lambda^2 y \exp(-\lambda y), y \geq 0,$ and $f(y) = 0$ otherwise.

(c) Suppose $\lambda = 1$. Let $c > 0$. Show that the density of $X_1 | X_1 > c$ is $\exp(-x) / \{1 - \exp(-c)\}, x > c$ and 0 otherwise.

(d) Suppose $\lambda = 1$. Let $c > 0$. Find the $E(X_1 | X_1 > c)$. 