Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master’s level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

**NOTATION**: We denote by $\mathbb{D}$ the open unit disc, i.e. $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$, and by $\gamma$ its boundary, traversed once counterclockwise.

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1. (a) Suppose $f$ is an entire function whose real part $u(x, y) = \text{Re}(f(x+iy))$ is a polynomial in $x, y$. Prove that $f(z)$ is a polynomial in $z$.

   (b) Suppose $f$ is an entire function whose real part $u(x, y) = \text{Re}(f(x+iy))$ is bounded above. Prove that $f$ is constant.

2. Suppose $\lambda$ is a real number satisfying $\lambda > 1$ and $f(z) = ze^{\lambda z} - 1$. Prove that $f(z)$ has a unique root in the unit disc $\mathbb{D}$ and that this root is a positive real number.

3. Suppose $f$ is holomorphic on the region $A = \{z \in \mathbb{C} : 0 < |z| < 2\}$, and that for all $n \geq 0$,
   \[ \int_{\gamma} z^n f(z) dz = 0, \]
   where $\gamma$ is the unit circle traversed once counterclockwise. Show that $f$ has a removable singularity at 0.

4. (a) For which $z$ in $\mathbb{C}$ does $\sum_{n=1}^{\infty} \frac{z^n}{1 + z^{2n}}$ converge?

   (b) At which $z$ in $\mathbb{C}$ is the sum $f(z)$ of this series holomorphic?

5. Write down a conformal map that takes the “right-half” of the unit disc, namely $R = \{z \in \mathbb{D} : \text{Re}(z) > 0\}$, onto the unit disc $\mathbb{D}$. 

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(6) For each of the following statements, if the statement is true, give a proof; if it is false, demonstrate this by giving a counterexample.
(a) If $f$ is holomorphic on a bounded connected open set $R \subset \mathbb{C}$ and has infinitely many zeros $z_1, z_2, z_3, \ldots$ in $R$, then $f$ is identically 0 on $R$.
(b) If $f, g$ are non-vanishing holomorphic functions on the open unit disc $\mathbb{D}$, satisfying
$$\frac{f'(1/n)}{f(1/n)} = \frac{g'(1/n)}{g(1/n)}, \quad n = 1, 2, 3, \ldots$$
then there exists a non-zero constant $c$ such that $f(z) = c g(z)$ for all $z \in \mathbb{D}$.

(7) Suppose $f$ is holomorphic and bounded on the region
$$A = \{ z \in \mathbb{C} : \frac{1}{2} < |z + i| \}$$
and is real on the real interval $(-1, 1) = \{ z \in \mathbb{R} : -1 < z < 1 \}$. State the Schwartz reflection principle and use it to prove that $f$ is constant.

(8) Let $f$ be holomorphic on the open unit disk $\mathbb{D}$ and let $d$ be the diameter of $f(\mathbb{D})$, that is, $d = \sup \{ |f(z_1) - f(z_2)| : z_1, z_2 \in \mathbb{D} \}$. Prove that
$$|f'(0)| \leq \frac{1}{2} d.$$  
Hint: Consider the function $g(z) = f(z) - f(-z)$.

(9) Use contour integration to evaluate, for real $\alpha > 0$, the improper integral
$$\int_{-\infty}^{\infty} \frac{\cos(\alpha x)}{1 + x^2} \, dx.$$  
Be sure to justify all your steps.

(10) Calculate
$$\int_{\gamma} \frac{(z^2 - 1)^2}{z^2 (z^2 + 4z + 1)} \, dz$$
where $\gamma$ is the unit circle traversed once in the counterclockwise direction. Be sure to justify all your steps.