BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 10 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Let V be the vector subspace of $\mathbf{R}^4$ generated by the vectors

$$v_1 = (1, 2, -1, -2),\ \ v_2 = (3, 1, 1, 1),\ \ v_3 = (-1, 0, 1, -1),$$

and W the subspace generated by

$$w_1 = (2, 5, -6, -5),\ \ w_2 = (-1, 2, -7, 3).$$

Find the dimension and a basis for each of $V \cap W$ and $V + W$.

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2. It is a fact that $M_n(\mathbf{R})$, the set of $n \times n$ matrices with real coefficients, is a real vector space. Given any $A \in M_n(\mathbf{R})$, show that the matrices

$$I_n, A, A^2, A^3, \dots,$$

spans a subspace of $M_n(\mathbf{R})$ of dimension $\leq n$.

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3. Given complex numbers a, b, c, d, find sufficient and necessary conditions for the matrix

$$\begin{pmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{pmatrix}$$

to be diagonalizable.

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4. Let A be an $n \times n$ real matrix. Prove that if A is orthogonal, symmetric and positive definite, then A is the identity.

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5. Determine the signature of the bilinear form on $\mathbf{R}^3$ with matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

Part II. Advanced Calculus

1. Give an example of a sequence of discontinuous functions on $[0, 1]$ converging uniformly to a continuous function. Justify your reasoning.

2(a) Show that the function

$$f(x) = \begin{cases} (1+x)^{1/x} & \text{if } x \neq 0, \\ e & \text{if } x = 0 \end{cases}$$

is infinitely differentiable on the entire real line.

(b) Determine the first three non-zero terms of the Taylor series of f expanded at the origin.

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3. Determine all points at which the surfaces $x^2 + y^2 + z^2 = 3$ and $x^3 + y^3 + z^3 = 3$ share the same tangent line.

Note: Any such point must of course lie on both surfaces.

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4. Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers. If it converges absolutely to a finite number S, show that any rearrangement of this series also converges to S.

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5. Evaluate

$$\int_C (y + \sin x)dx + (z^2 + \cos y)dy + x^3 dz$$

where C is the curve given parametrically by

$$\mathbf{r}(t) = \langle \sin t, \cos t, \sin 2t \rangle, \qquad 0 \leq t \leq 2\pi.$$

Hint: C lies on the surface $z = 2xy$.
