University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
January 10, 2011

Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1. Prove or disprove the following statements:

a) Let M be an n-dimensional compact manifold and $f\colon M \to \mathbb{R}^n$ a smooth map. Then f has a critical point.

b) Let M and N be smooth manifolds and $f\colon M \to N$ a smooth map. Suppose there exists a hypersurface $Z \subset M$ such that the restriction of f to $M \setminus Z$ is a submersion. Then f is a submersion.

Problem 2. **a)** Let α_i, $i = 1, 2, \cdots, m$, be C^∞ 1-forms on an n-dimensional manifold, $n \geq m$, which are linearly independent pointwise. Show that for any 1-forms β_i, $i = 1, 2, \cdots, m$, if

$$\sum_{i=1}^{m} \alpha_i \wedge \beta_i = 0,$$

then each β_i lies in the span of $\alpha_1, \cdots, \alpha_m$.

b) Let α be a C^∞ 1-form on a smooth manifold M and suppose there exists a nowhere zero C^∞ function f on M such that $d(f\alpha) = 0$. Prove that $\alpha \wedge d\alpha = 0$.

Problem 3. Let M, N, and X be smooth manifolds and $f\colon M \to X$ and $g\colon N \to X$ smooth maps. Let

$$Z \ := \ \{(p, q) \in M \times N : f(p) = g(q)\}.$$

Suppose that for each $(p, q) \in Z$,

$$f_{*,p}(T_p(M)) + g_{*,q}(T_q(N)) = T_{f(p)}(X).$$

Prove that Z is an embedded submanifold of $M \times N$.

Problem 4. Let M be a smooth manifold and $X \in \mathcal{X}(M)$, a C^∞ vector field on M. We define the support of the vector field X as the closure of the set

$$\{p \in M : X(p) \neq 0\}.$$

Prove that a C^∞ vector field X with compact support is complete. Give a counterexample that shows that the statement is false without the assumption that X has compact support.

Problem 5. We define a connection on $\mathbb{R}^3$ by setting

$$\nabla_{\partial_i}(\partial_j) \;:=\; \sum_{k=1}^{3} \Gamma_{ij}^k \, \partial_k \;; \quad \partial_i = \partial/\partial x_i,$$

where

$$\Gamma_{12}^3 = \Gamma_{23}^1 = \Gamma_{31}^2 = 1, \quad \Gamma_{21}^3 = \Gamma_{32}^1 = \Gamma_{13}^2 = -1,$$

and all other Christoffel symbols are zero.

a) Show that this connection is compatible with the Euclidean metric.
b) Determine the geodesics of this connection.
c) Is ∇ the Riemannian (Levi-Civita) connection for the Euclidean metric? Explain your answer.

Problem 6. We denote by G the set of 3×3 real matrices:

$$G \;:=\; \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & z & 0 \\ y & 0 & z \end{pmatrix} : z > 0 \right\}.$$

a) Show that G is a Lie subgroup of $GL(3, \mathbb{R})$.
b) Describe the Lie algebra of G as a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$.
c) Find the one-parameter subgroups of G.

Problem 7. Give examples of the following or prove that they cannot exist:

a) A diffeomorphism $F\colon S^4 \to S^2 \times S^2$.
b) A non-orientable embedded submanifold of an orientable manifold.
c) Vector bundles E, F and G over a manifold M such that

$$E \oplus F \cong E \oplus G$$

but $F \not\cong G$.

Problem 8. Let M be an open set of $\mathbb{R}^2$ with the Riemannian metric whose matrix in the standard frame ∂_x, ∂_y is given by:

$$\begin{pmatrix} 1 & 0 \\ 0 & \lambda^2(x, y) \end{pmatrix},$$

where λ is a nowhere-zero, smooth function on M.

a) Compute $\mathrm{grad}(f)$, for $f \in C^\infty(M)$ in terms of the standard frame ∂_x, ∂_y.
b) Compute the Gaussian curvature of M.
c) Find the differential equations for a geodesic in M.