Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Prove or disprove the following statements:

a) Let $M$ be an $n$-dimensional compact manifold and $f: M \to \mathbb{R}^n$ a smooth map. Then $f$ has a critical point.

b) Let $M$ and $N$ be smooth manifolds and $f: M \to N$ a smooth map. Suppose there exists a hypersurface $Z \subset M$ such that the restriction of $f$ to $M \setminus Z$ is a submersion. Then $f$ is a submersion.

Problem 2. a) Let $\alpha_i, i = 1, 2, \cdots, m$, be $C^\infty$ 1-forms on an $n$-dimensional manifold, $n \geq m$, which are linearly independent pointwise. Show that for any 1-forms $\beta_i, i = 1, 2, \cdots, m$, if

$$\sum_{i=1}^m \alpha_i \wedge \beta_i = 0,$$

then each $\beta_i$ lies in the span of $\alpha_1, \cdots, \alpha_m$.

b) Let $\alpha$ be a $C^\infty$ 1-form on a smooth manifold $M$ and suppose there exists a nowhere zero $C^\infty$ function $f$ on $M$ such that $d(f \alpha) = 0$. Prove that $\alpha \wedge d\alpha = 0$.

Problem 3. Let $M, N$, and $X$ be smooth manifolds and $f: M \to X$ and $g: N \to X$ smooth maps. Let

$Z := \{(p, q) \in M \times N : f(p) = g(q)\}$.

Suppose that for each $(p, q) \in Z$,

$$f_{*p}(T_p(M)) + g_{*q}(T_q(N)) = T_{f(p)}(X).$$

Prove that $Z$ is an embedded submanifold of $M \times N$.

Problem 4. Let $M$ be a smooth manifold and $X \in \mathcal{X}(M)$, a $C^\infty$ vector field on $M$. We define the support of the vector field $X$ as the closure of the set

$$\{p \in M : X(p) \neq 0\}.$$

Prove that a $C^\infty$ vector field $X$ with compact support is complete. Give a counterexample that shows that the statement is false without the assumption that $X$ has compact support.
Problem 5. We define a connection on $\mathbb{R}^3$ by setting
\[ \nabla_{\partial_i}(\partial_j) := \sum_{k=1}^{3} \Gamma^k_{ij} \partial_k; \quad \partial_i = \partial/\partial x_i, \]
where
\[ \Gamma^3_{12} = \Gamma^1_{23} = \Gamma^2_{31} = 1, \quad \Gamma^3_{21} = \Gamma^1_{32} = \Gamma^2_{13} = -1, \]
and all other Christoffel symbols are zero.

a) Show that this connection is compatible with the Euclidean metric.
b) Determine the geodesics of this connection.
c) Is $\nabla$ the Riemannian (Levi-Civita) connection for the Euclidean metric? Explain your answer.

Problem 6. We denote by $G$ the set of $3 \times 3$ real matrices:
\[ G := \left\{ \begin{pmatrix} 1 & 0 & 0 \\ x & z & 0 \\ y & 0 & z \end{pmatrix} : z > 0 \right\}. \]

a) Show that $G$ is a Lie subgroup of $GL(3, \mathbb{R})$.
b) Describe the Lie algebra of $G$ as a subalgebra of $\mathfrak{gl}(3, \mathbb{R})$.
c) Find the one-parameter subgroups of $G$.

Problem 7. Give examples of the following or prove that they cannot exist:
a) A diffeomorphism $F : S^4 \to S^2 \times S^2$.
b) A non-orientable embedded submanifold of an orientable manifold.
c) Vector bundles $E$, $F$ and $G$ over a manifold $M$ such that
\[ E \oplus F \cong E \oplus G \]
but $F \not\cong G$.

Problem 8. Let $M$ be an open set of $\mathbb{R}^2$ with the Riemannian metric whose matrix in the standard frame $\partial_x, \partial_y$ is given by:
\[ \begin{pmatrix} 1 & 0 \\ 0 & \lambda^2(x,y) \end{pmatrix}, \]
where $\lambda$ is a nowhere-zero, smooth function on $M$.
a) Compute $\text{grad}(f)$, for $f \in C^\infty(M)$ in terms of the standard frame $\partial_x, \partial_y$.
b) Compute the Gaussian curvature of $M$.
c) Find the differential equations for a geodesic in $M$. 