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ADVANCED EXAM — DIFFERENTIAL EQUATIONS
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Do five of the following seven problems. All problems carry equal weight. Passing level: 75% with at least three substantially complete solutions.

1. (a) Let B be a real, 2×2 matrix with eigenvalues $\beta_1 < -1$ and $\beta_2 > 0$, and with assoicated eigenvectors v_1 and v_2. Let A be the 4×4 block matrix
$$A = \begin{pmatrix} -2I_2 & B \\ I_2 & 0_2 \end{pmatrix},$$
where I_2 and O_2 are the 2×2 identity and zero matrices, respectivelty. Determine the (real) Jordan form of the exponential matrix e^{tA}, i.e determine an expression of the form
$$e^{tA} = Ve^{tJ}V^{-1}, \tag{1}$$
where J is a 4×4 matrix in real Jordan form and V is a suitable invertible 4×4 matrix.

   [A complete solution should give a detailed description of how the matrices V, J, and e^{tJ} are derived from the given information about B and the structure of A. You need not calculate V^{-1} and your answer should be left in factorized form without multiplying out the various matrix products in (1).]

   (b) Describe the stable and unstable subspaces of the system $y' = Ay$.

2. (a) Consider the damped and driven wave equation
$$\frac{\partial^2 u}{\partial t^2} + 2\gamma\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \cos t \,\sin x\,, \qquad \text{on } 0 < x < \pi\,,$$
with the boundary conditions $u(0,t) = u(\pi,t) = 0$. Assume that $0 < \gamma < 1$. Formulate the initial value problem for this PDE and solve it explicitly using the Fourier method.

   (b) Determine the asymptotic behavior as $t \to +\infty$ of a general solution $u(x,t)$ to the initial value problem in (a).

3. Consider the system

$$\frac{dx}{dt} = Ax + q(x), \tag{2}$$

where A is a real $n \times n$ matrix with distinct negative eigenvalues $\lambda_i < \lambda_{i+1}$ for $1 \leq i \leq n-1$, and $q : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth, real-valued vector field satisfying an estimate of the form

$$|q(x)| \leq K|x|^2$$

for all $x \in \mathbb{R}^n$ and some positive constant K; $|x| = (\sum_{i=1}^n x_i^2)^{\frac{1}{2}}$.
Show that there exist $r_o > 0$ and an invertible, real $n \times n$ matrix V such that if $0 < r \leq r_o$ the set

$$\Sigma_r = \{x = Vy : \quad |y| \leq r\}$$

is a positively invariant set for (2), i.e. $x(0) \in \Sigma_r$ implies that $x(t) \in \Sigma_r$ for all $t \geq 0$.

[Your discussion should determine a value of r_o in terms of K and various data obtained from the matrix A.]

4. Consider the nonlinear parabolic PDE,

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - f'(u) \qquad (-\infty < x < +\infty\,, \quad t > 0)$$

where $f(z)$ is a smooth convex function of $z \in \mathbb{R}$ with $f'(0) = 0$. Prove the uniqueness of solutions to the initial value problem in the following sense:
If $u_1(x,t)$ and $u_2(x,t)$ are classical (sufficiently smooth) solutions that vanish (sufficiently rapidly) as $x \to \pm\infty$, and $u_1(x,0) \equiv u_2(x,0)$ identically in x, then $u_1(x,t) \equiv u_2(x,t)$ for all $t > 0$.

Hint: Use an "energy method" on the difference $w = u_1 - u_2$.

5. Consider the planar system

$$\begin{aligned} \frac{dx}{dt} &= \frac{4}{\pi}\arctan x - y \\ \frac{dy}{dt} &= y - x^3. \end{aligned} \qquad (3)$$

(a) Determine all rest points and the local behavior of solutions in a small neighborhoods of each rest point.
(b) Determine whether the system admits any heteroclinic, homoclinic, or periodic solutions; give a proof of the existence of any of these solutions that do occur, and use this analysis to sketch the global phase portrait of (3).

6. (a) Define the Sobolev space $H^1(\Omega)$, for a bounded domain $\Omega \subset \mathbb{R}^n$ with smooth boundary $\partial\Omega$. Show that if $\alpha \neq 0$, given any $f \in L^2(\Omega)$, there exists a unique weak solution $u \in H^1(\Omega)$ to the elliptic boundary value problem

$$-\Delta u + \alpha^2 u = f(x) \quad \text{in } \Omega, \qquad \frac{\partial u}{\partial N} = 0 \quad \text{on } \partial\Omega\,,$$

where N denotes the outward unit normal on Ω.
(b) Explain how the boundary condition on $\partial\Omega$ is incorporated in the weak formulation used in part (b).

7. (a) Determine an expression for the general solution $y(t)$ of the linear system

$$\frac{dy}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & 2 \end{pmatrix} y, \tag{4}$$

as a linear combination of expressions involving the eigenvectors and eigenvalues of the coefficient matrix of the system

(b) If $y(t)$ is a solution of (4), define $x(t) = (p(t), q(t))$ by

$$p(t) = \frac{y_2(t)}{y_1(t)}, \quad q(t) = \frac{y_3(t)}{y_1(t)}.$$

Show that $x(t)$ is the solution of a (nonlinear) system two autonomous equations

$$\frac{dx}{dt} = f(x) \tag{5}$$

for some vector field $f : \mathbb{R}^2 \to \mathbb{R}^2$.

(c) Describe how the growth and decay properties of various solutions $y(t)$ of the linear system (4) determine the global phase plane of solutions $x(t)$ of the system (5). In particular, show how the system (4) for $y(t)$ determines all rest points and all heteroclinic solutions connecting pairs of rest points in the phase plane of (5).

Hint. The behavior of the span, $\{cy(t) : \; c \in \mathbb{R}\}$, of a solution $y(t)$ of (4) determines the trajectory of a single solution $x(t)$ of (5).