Work all problems. Show your work; explain your answers; state theorems used whenever possible.

1. A gene has two possible forms (alleles): A and a. Thus there are three possible genotypes: AA, aA, and aa. Number them 1, 2, and 3, respectively. Assume that their proportions in the population are \( p^2 \), \( 2pq \), and \( q^2 \), respectively (\( q = 1 - p \)).

For a family with a father, a mother, and one child, let the random variables \( F, M, \) and \( C \) denote the genotypes of the father, mother, and child, respectively. For example, \( F \) is either 1, 2, or 3, according to the genotype of the father. Assume that \( F \) is independent of \( M \), i.e. that the population mates randomly, and that the conditional distribution of \( C \) given \((F, M)\) is determined by the familiar rules of genetics. (Children inherit one gene from each parent; each parent’s gene has probability 0.5 of being chosen; the mother’s contribution is independent of the father’s contribution.) Let \( p_{ik} = \Pr[C = k | M = i] \), the conditional probability that the child is of type \( k \) given that the mother (or father) is of type \( i \). Compute the nine probabilities \( p_{ik} \) in terms of \( p \) and \( q \).

2. Let \( \vec{Y} \) have a trivariate Gaussian distribution with mean vector \( \vec{\mu} \) and covariance matrix \( \Sigma \), where

\[
\vec{Y} = \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}, \quad \vec{\mu} = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix}.
\]

(a) For which values of \( \rho \) are \( Y_1 + Y_2 + Y_3 \) and \( Y_1 - Y_2 - Y_3 \) statistically independent?

(b) What is the distribution of \( Y_1 + Y_2 + Y_3 \), including its name and associated parameters.

3. Suppose that \( X \) is a random variable with density \((3x + 1)/8\) on the interval \((0, 2)\). Let \( Y \) be the area of a circle of radius \( X \). Find the density of \( Y \).

4. (a) A continuous random variable \( Y \) takes values on the interval \((0, \infty)\). Show \( E[Y] = \int_0^\infty \Pr[Y \geq y] \, dy \). Hint: you may use the fact that \( y = \int_0^y \, dz \).

(b) A discrete random variable \( X \) takes values on the positive integers 1, 2, . . . . Show \( E[X] = \sum_{x=1}^{\infty} \Pr[X \geq x] \).

5. A family of densities is called a \textit{univariate natural exponential} family if, for some function \( A(\theta) \), the density of \( X \) given \( \theta \) can be expressed as

\[
p(x | \theta) = h(x) e^{\theta x - A(\theta)}.
\]

Suppose that \( X \) has such a density.

(a) Show that the moment generating function \( M_X(\theta)(t) = E[e^{tX}] \) is \( e^{\theta(A(t)+t) - A(\theta)} \).

(b) Show that \( E[X] = A'(\theta) \).