Do five of the following problems. All problems carry equal weight.

Passing level:

**Masters:** 60% with at least two substantially correct

**PhD:** 75% with at least three substantially correct.

1. Consider \( f(x) = 2 - x + x^2 - x^3 \). Let \( p_2(x) \) denote the second degree polynomial interpolation of \( f(x) \) at \(-1, 0, 1\).

   (a) Find \( p_2(x) \).

   (b) Compute the \( L_\infty \) error of \( p_2(x) \) on the domain \([-1, 1]\).

2. Suppose that the function \( f(x) \) has enough regularity and \( a \) is a root of \( f(x) \). Find the order of convergence of the Newton’s method for the root \( a \), assuming that the initial guess is sufficiently close to \( a \). If the convergence is of order one, then give the rate of convergence (Hint: Make the appropriate assumptions on \( f'(a), f''(a), ... \)).

3. Find the Gauss-Lobatto like quadrature

\[
\int_{-1}^{1} f(x) dx \approx \omega_1 f(-1) + \omega_2 f(x_0) + \omega_3 f(1)
\]

with the highest possible degree of precision.

4. For function \( \sin(\pi x) \),

   (a) Find the value of \( a \) which solves the following optimization problem:

   \[
   \min_a \int_{-1}^{1} (\sin(\pi x) - ax)^2 \, dx
   \]

   (b) Let \( \hat{f}(x) \) be a polynomial with degree less than or equal to \( n > 1 \), which solves the minimization problem:

   \[
   \min_{p(x) \in P_n(x)} \int_{-1}^{1} (\sin(\pi x) - p(x))^2 \, dx
   \]

   Prove that \( \hat{f}(x) \) is an odd function.

5. Consider the ordinary differential equation \( \frac{dy}{dt} = 0.1y \)

   (a) What is the order of the scheme: \( y^{n+1} - y^n = 0.1 \Delta t y^n? \) Derive the local truncation error.

   (b) Derive a 10th order scheme for the above equation.
6. (a) Write down the Jacobi and Gauss-Siedel methods for the system \( Ax = b \) where 
\[
A = \begin{pmatrix}
3 & 1 & 0 \\
2 & 2 & 0 \\
0 & 2 & 3 \\
\end{pmatrix}
\]
and 
\[
b = \begin{pmatrix}
6 \\
1 \\
1 \\
\end{pmatrix}
\]
(b) Prove or disprove that the Jacobi method for the system above converges for any initial guess.

7. Consider the fixed point iteration 
\[
x_{n+1} = \phi(x_n), \quad n = 0, 1, 2, ...
\]
where \( \phi(x) = Ax + Bx^2 + Cx^3 \).

(a) Given a positive number \( \alpha \), determine the constants \( A, B, C \) such that the iteration converges locally to \( 1/\alpha \) with order 3 (This will give a cubically convergent method for computing the reciprocal \( 1/\alpha \) which uses only addition, subtraction and multiplication).

(b) Determine the maximal possible interval in which the initial guess \( x_0 \) can lie in order that the iteration \( x_n \) converges to \( \frac{1}{\alpha} \).