Department of Mathematics and Statistics
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Basic Exam — Complex Analysis

January 11, 2010

Provide solution for *eight* of the following *ten* problems. Indicate clearly which problems you want graded. Each problem is worth 10 points.

Passing standard: To pass at the Master's level it is sufficient to have 45 points with 3 essentially correct solutions. To pass at the Ph. D level it is sufficient to have 55 points with 4 essentially correct solutions.

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1. Let f be a holomorphic automorphism of $\mathbb{C}^* = \mathbb{C} - \{0\}$. Show that there is a number $a \in \mathbb{C}^*$ such that either $f(z) = az$ or $f(z) = az^{-1}$.

2. Consider the polynomial $P(z) = 2z^5 - 2z^3 + z - 7$.

   (a) Show that it has no zeros in the disc $|z| \leq 1$.

   (b) Find the number of its zeros in the disc $|z| \leq 2$.

   (c) How many zeros does it have in the annulus $100 < |z| < 1000$?

3. Carefully state and prove the Liouville theorem on entire functions.

4. Suppose that $f(z)$ is continuous on the unit disc $D = \{ z \in \mathbb{C} : |z| < 1 \}$ and that it is holomorphic on $D - \mathbb{R} = \{ z \in \mathbb{C} : |z| < 1 \text{ and } z \notin \mathbb{R} \}$. Prove that f is then holomorphic on all of D.

5. Find the Laurent expansion of

$$f(z) = \frac{z^2 + z - 1}{(z+2)(z-1)^2}$$

   in an annulus (centered at the origin) that contains $\sqrt{3}$. What is the largest possible annulus on which this expansion is valid?

6. Calculate the following integral using residues

$$\int_0^\infty \frac{x \sin(x)}{x^2 + 1}\, dx.$$

   Justify your calculation.

7. Calculate the following integral using residues

$$\int_{-\infty}^{+\infty} \frac{x}{(x^2 - 2x + 5)^2}\, dx.$$

Justify your calculation.

8. If $f \neq 0$ is an entire function and $|f(z)| \leq \ |z^3 + z^2 + z + 1|$ for all $z \in \mathbb{C}$, find

$$\frac{f(1) + f(-1)}{2f(0)}.$$

9. Let $f\colon D \to S$ be a conformal bijection between the disc $D = \{\, z : |z| < 1 \,\}$ and the square $S = \{\, z + iy : |x| < 1 \text{ and } |y| < 1 \,\}$. Prove that if $f(0) = 0$ then $-f(z) = f(-z)$.

10. Find a conformal map that maps the semi-disc

$$S = \{\, z : |z| < 1 \text{ and } \operatorname{Re}(z) > 0 \,\}$$

to the first quadrant

$$Q = \{\, z : \operatorname{Re}(z) > 0 \text{ and } \operatorname{Im}(z) > 0 \,\}$$

and sends the points

$$A = i, \qquad B = -i, \qquad C = 1$$

to

$$A' = \infty, \qquad B' = 0, \qquad C' = 1,$$

respectively.