Do 7 of the following 9 problems.

Passing Standard: For Master’s level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Find non-zero vectors $\bar{x}, \bar{y}, \bar{b} \in \mathbb{R}^3$ such that $A\bar{x} = \bar{b} = B\bar{y}$, where
   \[
   A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.
   \]

2. Let
   \[
   V_1 \xleftarrow{f_1} V_2 \\
   \uparrow f_4 \quad \downarrow f_2 \\
   V_4 \xleftarrow{f_3} V_3
   \]
   be a collection of linear maps between finite dimensional real vector spaces. Suppose that
   \[
   \text{im}(f_1) = \text{Null}(f_2), \quad \text{im}(f_2) = \text{Null}(f_3), \quad \text{im}(f_3) = \text{Null}(f_4), \quad \text{im}(f_4) = \text{Null}(f_1).
   \]
   Show that
   \[
   \dim V_1 + \dim V_3 = \dim V_2 + \dim V_4.
   \]

3. Let $V$ be the space of $n \times n$ real matrices.
   (a) Show that
   \[
   \langle A, B \rangle := \text{trace}((A^t)B)
   \]
   is a positive definite, symmetric inner product on $V$.
   (b) Find an orthonormal basis with respect to this inner product for $n = 2$.

4. Let $A$ be an $n \times n$ matrix over the complex numbers. Define
   \[
   \rho(A) := \max\{ |\lambda| : \lambda \in \mathbb{C} \text{ is an eigenvalue of } A \}.
   \]
   Show that if $\rho(A) < 1$, then $\lim_{m \to \infty} A^m = 0$.
   Note: the eigenvalues of $A$ need not be distinct.
Part II. Advanced Calculus

1. Find every point on the surface of the ellipsoid $x^2 + 4y^2 + 9z^2 = 16$ at which the normal line at that point passes through the origin.

Note: Recall that the normal line to a surface at a point $P$ is a line through $P$ which is perpendicular to the surface.

2. Fix two positive real numbers $a_0, b_0$, and define two sequences as follow:

$$a_{n+1} := \frac{a_n + b_n}{2}, \quad b_{n+1} := \sqrt{a_nb_n}, \quad n = 0, 1, 2, \ldots$$

(a) Suppose $a_0 \geq b_0$, show that

$$a_0 \geq a_1 \geq \cdots a_n \geq a_{n+1} \geq \cdots \geq b_{m+1} \geq b_m \geq \cdots \geq b_1 \geq b_0.$$ 

You can make use of the arithmetic-geometric inequality: For any positive real numbers $x, y$, we have $\frac{x + y}{2} \geq \sqrt{xy}$.

(b) For $n = 0, 1, 2, \ldots$, show that $0 \leq a_n - b_n \leq \frac{a_0 - b_0}{2^n}$.

(c) Show that $\lim_{n \to \infty} a_n$ and $\lim_{n \to \infty} b_n$ both exist, and that the two limits are equal. This common value is called the Arithmetic-Geometric mean of $a_0$ and $b_0$ (first investigated by Gauss).

3. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval of $\mathbb{R}$, but it does not converge absolutely for any fixed value of $x$.

4. Denote by $C$ the curve of intersection of the plane $z = ax + by$ with the cylinder $x^2 + y^2 = 1$, with counterclockwise orientation when viewed from the top.

(a) Compute the line integral $\int_C (ydx + (z - x)dy - ydz)$ directly by parametrizing the curve $C$.

(b) Using Stoke’s theorem, deduce a formula for the area of the region bounded by $C$ in the given plane, when $a \neq 1$.

5. Determine the point on the surface

$$S := \text{the portion of the surface } xyz^2 = 1 \text{ that lies in the first octant that is closest to the origin}.$$