BASIC EXAM – LINEAR ALGEBRA/ADVANCED CALCULUS
UNIVERSITY OF MASSACHUSETTS, AMHERST
DEPARTMENT OF MATHEMATICS AND STATISTICS
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Do 7 of the following 9 problems.

Passing Standard: For Master's level, 60% with three questions essentially complete (including at least one from each part). For Ph. D. level, 75% with two questions from each part essentially complete.

Show your work!

Part I. Linear Algebra

1. Find *non-zero* vectors $\vec{x}, \vec{y}, \vec{b} \in \mathbf{R}^3$ such that $A\vec{x} = \vec{b} = B\vec{y}$, where

$$A = \begin{pmatrix} 0 & 1 & 2 \\ 1 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{pmatrix}.$$

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2. Let

$$\begin{array}{ccc} V_1 & \xrightarrow{f_1} & V_2 \\ {\scriptstyle f_4}\uparrow & & \downarrow{\scriptstyle f_2} \\ V_4 & \xleftarrow{f_3} & V_3 \end{array}$$

be a collection of linear maps between finite dimensional real vector spaces. Suppose that

$$\operatorname{im}(f_1) = \operatorname{Null}(f_2),\ \operatorname{im}(f_2) = \operatorname{Null}(f_3),\ \operatorname{im}(f_3) = \operatorname{Null}(f_4),\ \operatorname{im}(f_4) = \operatorname{Null}(f_1).$$

Show that

$$\dim V_1 + \dim V_3 = \dim V_2 + \dim V_4.$$

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3. Let V be the space of $n \times n$ real matrices.

(a) Show that

$$\langle A, B\rangle := \operatorname{trace}((A^t)B)$$

is a *positive definite*, *symmetric inner product* on V.

(b) Find an orthonormal basis with respect to this inner product for $n = 2$.

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4. Let A be an $n \times n$ matrix over the complex numbers. Define

$$\rho(A) := \max\{|\lambda| : \lambda \in \mathbf{C} \text{ is an eigenvalue of } A\}.$$

Show that if $\rho(A) < 1$, then $\lim_{m\to\infty} A^m = 0$.

Note: the eigenvalues of A need *not* be distinct.

Part II. Advanced Calculus

1. Find every point on the surface of the ellipsoid $x^2 + 4y^2 + 9z^2 = 16$ at which the normal line at that point passes through the origin.

Note: Recall that the normal line to a surface at a point P is a line through P which is perpendicular to the surface.

2. Fix two positive real numbers a_0, b_0, and define two sequences as follow:

$$a_{n+1} := \frac{a_n + b_n}{2}, \;\; b_{n+1} := \sqrt{a_n b_n}, \;\; n = 0, 1, 2, \ldots$$

(a) Suppose $a_0 \geq b_0$, show that

$$a_0 \geq a_1 \geq \cdots a_n \geq a_{n+1} \geq \cdots \geq b_{m+1} \geq b_m \geq \cdots \geq b_1 \geq b_0.$$

You can make use of the *arithmetic-geometric inequality*: *For any positive real numbers x, y, we have* $\dfrac{x+y}{2} \geq \sqrt{xy}$.

(b) For $n = 0, 1, 2, \ldots$, show that $0 \leq a_n - b_n \leq \dfrac{a_0 - b_0}{2^n}$.

(c) Show that $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ both exist, and that the two limits are equal. This common value is called the *Arithmetic-Geometric mean* of a_0 and b_0 (first investigated by Gauss).

3. Prove that the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{x^2 + n}{n^2}$$

converges uniformly in every bounded interval of $\mathbf{R}$, but it does not converge absolutely for any fixed value of x.

4. Denote by C the curve of intersection of the plane $z = ax + by$ with the cylinder $x^2 + y^2 = 1$, with counterclock-wise orientation when viewed from the top.

(a) Compute the line integral $\displaystyle\int_C \big(y dx + (z - x) dy - y dz\big)$ directly by parametrizing the curve C.

(b) Using Stoke's theorem, deduce a formula for the area of the region bounded by C in the given plane, when $a \neq 1$.

5. Determine the point on the surface

$$S := \text{the portion of the surface } xyz^2 = 1 \text{ that lies in the first octant}$$

that is closest to the origin.