Department of Mathematics and Statistics
Basic Probability Exam
January 2009

Work all problems. Show your work. Explain your answers. State the theorems used whenever possible. 60 points are needed to pass at the Masters level and 75 to pass at the Ph.D. level

1. Ecologists are studying salamanders in a forest. There are two types of forest. Type A is conducive to salamanders while type B is not. They are studying one forest but don't know which type it is. Types A and B are equally likely.

   During the study, they randomly sample quadrats. (A quadrat is a square-meter plot.) In each quadrat they count the number of salamanders. Some quadrats have poor salamander habitat. In those quadrats the number of salamanders is 0. Other quadrats have good salamander habitat. In those quadrats the number of salamanders is either 0, 1, 2, or 3, with probabilities 0.1, 0.3, 0.4, and 0.2, respectively. (Yes, there might be no salamanders in a quadrat with good habitat.) In a type A forest, the probability that a quadrat is good is 0.8 and the probability that it is poor is 0.2. In a type B forest the probability that a quadrat is good is 0.3 and the probability that it is poor is 0.7.

   (a) **4 pts** On average, what is the probability that a quadrat is good?

   (b) **5 pts** On average, what is the probability that a quadrat has 0 salamanders, 1 salamander, 2 salamanders, 3 salamanders?

   (c) **4 pts** The ecologists sample the first quadrat. It has 0 salamanders. What is the probability that the quadrat is good?

   (d) **4 pts** Given that the quadrat had 0 salamanders, what is the probability that the forest is type A?

   (e) **4 pts** Now the ecologists prepare to sample the second quadrat. Given the results from the first quadrat, what is the probability that the second quadrat is good?

   (f) **4 pts** Given the results from the first quadrat, what is the probability that they find no salamanders in the second quadrat?

2. A Poisson random variable with mean μ has the following p.d.f:

$$f(x) = \frac{e^{-\mu}\mu^x}{x!}, \text{for } x = 0, 1, 2, \ldots$$

(a) **11 pts** Let X be Poisson with mean μ. Compute the moment generating function of X. It is known that:

$$e^y = \sum_{k=0}^{\infty} \frac{y^k}{k!}.$$

(b) **7 pts** If $X_1, \ldots, X_n$ are independent Poisson variables with means $\mu_1, \ldots, \mu_n$, find the moment generating function of

$$Y = \sum_{k=1}^{n} X_k.$$

(c) **7 pts** What is the distribution of Y?

3. Let $X \sim \mathrm{N}(\mu, \sigma^2)$.

   (a) **12 pts** Show that the moment generating function of X is:

   $$M_X(t) = \exp(\mu t + \sigma^2 t^2/2)$$

   (b) **8 pts** Show that if $X_1 \sim \mathrm{N}(\mu_1, \sigma_1^2)$ and $X_2 \sim \mathrm{N}(\mu_2, \sigma_2^2)$ and X_1 and X_2 are independent, then $X_1 + X_2 \sim \mathrm{N}(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.

4. (a) Let (X_1, X_2) be distributed uniformly on the disk where $X_1^2 + X_2^2 \leq 1$. Let $R = \sqrt{X_1^2 + X_2^2}$ and $\Theta = \arctan(X_1/X_2)$. Hint: it may help to draw a picture.

   i. **1 pt** What is the joint density $p(x_1, x_2)$?

   ii. **4 pts** Are X_1 and X_2 independent? Explain.

   iii. **15 pts** Find the joint density $p(r, \theta)$.

   iv. **4 pts** Are R and Θ independent? Explain.

   (b) Let (X_1, X_2) be distributed uniformly on the square whose corners are $(1, 1)$, $(-1, 1)$, $(-1, -1)$, and $(1, -1)$. Let $R = \sqrt{X_1^2 + X_2^2}$ and $\Theta = \arctan(X_1/X_2)$.

   i. **1 pt** What is the joint density $p(x_1, x_2)$?

   ii. **1 pt** Are X_1 and X_2 independent? Explain.

   iii. **4 pts** Are R and Θ independent? Explain.