DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS AMHERST
BASIC NUMERIC ANALYSIS EXAM
JANUARY 2009

Do five of the following problems. All problems carry equal weight.
Passing level:

Masters: 60% with at least two substantially correcct
PhD: 75% with at least three substantially correct.

1. Suppose one were to use Newton's method to approximate $\sqrt{a}$, where $a > 0$, by finding the positive root of $f(x) = x^2 - a = 0$. Assume the initial guess x_0 satisfies $x_0 > 0$ and $x_0 \neq \sqrt{a}$. Prove the following:
   (a) $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{a}{x_n}\right)$
   (b) $x_{n+1}^2 - a = \left(\dfrac{x_n^2 - a}{2x_n}\right)^2$ for $n \geq 0$, and therefore $x_n > \sqrt{a}$ for all $a > 0$.
   (c) The iterates $\{x_n\}_{n=0}^{\infty}$ are a strictly decreasing sequence for $n \geq 1$. *Hint*: Consider the sign of $x_{n+1} - x_n$.

2. Let $f(x)$ be a step function defined on $[-1, 1]$ as follows,

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1]. \end{cases}$$

   (a) Find the interpolation polynomials of degree zero (denoted as $p_0(x)$), degree one (denoted as $p_1(x)$), and degree two (denoted as $p_2(x)$), for $f(x)$ on $[-1, 1]$. Suppose that equispaced interpolation points are used. To be more specific, the interpolation point sets are $\{0\}, \{-1, 1\}$, and $\{-1, 0, 1\}$ for $p_0(x), p_1(x), p_2(x)$ respectively.
   (b) Compute the *maximum* errors of these three interpolations.
   (c) Is the polynomial interpolation on equispaced points convergent for $f(x)$ in terms of the *maximum* error? Why? (Hint: Show the error is bounded from below.)

3. Again, consider the step function $f(x)$ defined on $[-1, 1]$ as follows,

$$f(x) = \begin{cases} 1, & x \in [-1, 0], \\ -1, & x \in (0, 1]. \end{cases}$$

   (a) Find the constant, linear, and quadratic least square approximate $p_0(x)$, $p_1(x)$, and $p_2(x)$ to $f(x)$ on $[-1, 1]$.
   (b) Compute the L^2 errors of these three least square approximates.

4. Derive the 2 node Gauss-Legendre quadrature formula

$$\int_{-1}^{1} f(x)\, dx \approx c_1 f(x_1) + c_2 f(x_2)\,.$$

So you must find x_1, x_2, c_1, c_2. Recall, the second degree Legendre polynomial is $P_2(x) = x^2 - 1/3$.

5. Consider the numeric scheme

$$\begin{aligned} k_1 &= f(x_n, y_n) \\ k_2 &= f(x_n + \frac{1}{2}h, y_n + \frac{1}{2}hk_1) \\ y_{n+1} &= y_n + hk_2 \end{aligned}$$

for the ODE $dy/dx = f(x, y)$. Show that is a second order scheme and compute the leading term of the truncation error.

6. Suppose A is a real $n \times n$ symmetric matrix, i.e. $A \in \mathbb{R}^{n\times n}$ and $A^T = A$.

   (a) Define what is means for A to be positive definite.

   (b) Suppose also that A is positive definite. Prove that the diagonal entries of A are all positive.

   (c) Suppose also that A is positive definite. Prove that the largest entry of A in absolute value lies on the diagonal.

7. When solving the one-dimensional heat equation with a second-order central difference scheme, an example resultant linear system after imposing boundary conditions takes the following form.

$$\begin{array}{rrrcr} -4x_1 & +x_2 & & = & 1 \\ x_1 & -4x_2 & +x_3 & = & -3 \\ & x_2 & -4x_3 & = & -3 \end{array}$$

   (a) Find the **LU** decomposition of the coefficient matrix of the above system.

   (b) Solve the system with the LU decomposition obtained in (a).