Department of Mathematics and Statistics
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Basic Exam - Complex Analysis

January 2009

Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master's level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Determine the Laurent series for the function $f(z) = \dfrac{2}{(z^2-3z+2)(z-3)}$ in the annulus $1 < |z| < 2$.

2. (a) Determine the number of zeroes of $z^5 - 2z^2 + z + 1$ in the disk $\{z : \ |z| < 10\}$.
   (b) Compute the integral
   $$\int_{\{|z|=10\}} \frac{3z^4+1}{z^5-2z^2+z+1} dz$$

3. Compute $\displaystyle\int_C \frac{z^7 e^{1/z}}{1-z^7} dz$ where C denotes the circle $\{|z| = 2\}$ traversed counter-clockwise.

4. Compute $\displaystyle\int_0^\infty \frac{x^2 dx}{(x^4+1)(x^2+1)}$. Justify your computation. In particular, prove all estimates.

5. Let C be a simply closed contour in a simply-connected domain D and f a meromorphic function on D, which is holomorphic along C. Prove that
   $$\frac{1}{2\pi i}\int_C \frac{f'(z)}{f(z)} dz$$
   is an integer, equal to the number of zeroes minus the number of poles enclosed by C, counted with multiplicity.

6. Find a one-to-one conformal map from the region $\{z \mid \mathrm{Re}(z) < 0,\ 0 < \mathrm{Im}(z) < \pi\}$ onto the open unit disk.

7. Determine the number of solutions of $\cos(z) = cz^n$ in the open unit disk for every positive integer n and constant c satisfying $|c| > e$.

8. Prove or disprove:
   (a) Any entire function is a limit of polynomials, uniform on bounded subsets.

(b) The image of the complex plane $\mathbb{C}$ under a non-constant entire function is dense in $\mathbb{C}$.

(c) If $g(z)$ is an entire function, $g(0) = 1$, $f(z) = \frac{g(z)}{z}$, $u(x,y) = \mathrm{Re}(f(x+iy))$, $v(x,y) = \mathrm{Im}(f(x+iy))$, and C is the circle of radius 1 centered at the origin, then

$$\int_C u(x,y)dx - v(x,y)dy = 0.$$

9. Let f be a one-to-one holomorphic map from a region Ω_1 onto a region Ω_2. Assume that the closure of the disc $D := \{z : |z - z_0| < \epsilon\}$ is contained in Ω_1. Prove that the inverse function $f^{-1} : f(D) \to D$ is given by the integral formula

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z) - \omega} \cdot z dz.$$

10. Let $U := \{z : |z| < 2 \text{ and } |z-1| > \frac{1}{2} \text{ and } |z+1| > \frac{1}{2}\}$, and f a holomorphic function on U. Recall that if γ_1 and γ_2 are closed chains in a region Ω in the complex plane, which are homologous in Ω, then $\int_{\gamma_1} g(z)dz = \int_{\gamma_2} g(z)dz$, for every function g holomorphic in Ω, by the general form of Cauchy's Theorem.

(a) Use Cauchy's Theorem to prove that there exists a decomposition $f(z) = f_1(z) + f_2(z) + f_3(z)$, for all $z \in U$, where

$$f_1(z) = \sum_{n=1}^{\infty} \alpha_n (z-1)^{-n},$$

$$f_2(z) = \sum_{n=1}^{\infty} \beta_n (z+1)^{-n},$$

$$f_3(z) = \sum_{n=0}^{\infty} \gamma_n z^n,$$

and each of the series converges absolutely in U and uniformly on compact subsets of U. *Hint: Express $f(z)$ as an integral over three circles (indicate the domain Ω chosen in the application of Cauchy's Theorem and provide a proof to any claim that two chains are homologous in Ω).*

(b) Prove that the above decomposition is unique. *Hint: Relate f_i to the Laurent series of f in suitable annuli contained in U.*