Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master’s level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Determine the Laurent series for the function \( f(z) = \frac{2}{(z^2 - 3z + 2)(z - 3)} \) in the annulus \( 1 < |z| < 2 \).

2. (a) Determine the number of zeroes of \( z^5 - 2z^2 + z + 1 \) in the disk \( \{ z : |z| < 10 \} \).
   (b) Compute the integral
   \[
   \int_{\{|z|=10\}} \frac{3z^4 + 1}{z^5 - 2z^2 + z + 1} \, dz
   \]

3. Compute \( \int_C \frac{z^7 e^{1/z}}{1 - z^7} \, dz \) where \( C \) denotes the circle \( \{|z| = 2\} \) traversed counterclockwise.

4. Compute \( \int_0^\infty \frac{x^2 \, dx}{(x^4 + 1)(x^2 + 1)} \). Justify your computation. In particular, prove all estimates.

5. Let \( C \) be a simply closed contour in a simply-connected domain \( D \) and \( f \) a meromorphic function on \( D \), which is holomorphic along \( C \). Prove that
   \[
   \frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} \, dz
   \]
   is an integer, equal to the number of zeroes minus the number of poles enclosed by \( C \), counted with multiplicity.

6. Find a one-to-one conformal map from the region \( \{ z \mid \text{Re}(z) < 0, \ 0 < \text{Im}(z) < \pi \} \) onto the open unit disk.

7. Determine the number of solutions of \( \cos(z) = cz^n \) in the open unit disk for every positive integer \( n \) and constant \( c \) satisfying \( |c| > e \).

8. Prove or disprove:
   (a) Any entire function is a limit of polynomials, uniform on bounded subsets.
(b) The image of the complex plane $\mathbb{C}$ under a non-constant entire function is dense in $\mathbb{C}$.

(c) If $g(z)$ is an entire function, $g(0) = 1$, $f(z) = \frac{g(z)}{z}$, $u(x, y) = \text{Re}(f(x + iy))$, $v(x, y) = \text{Im}(f(x + iy))$, and $C$ is the circle of radius 1 centered at the origin, then

$$\int_C u(x, y)dx - v(x, y)dy = 0.$$ 

9. Let $f$ be a one-to-one holomorphic map from a region $\Omega_1$ onto a region $\Omega_2$. Assume that the closure of the disc $D := \{z : |z - z_0| < \epsilon\}$ is contained in $\Omega_1$. Prove that the inverse function $f^{-1} : f(D) \to D$ is given by the integral formula

$$f^{-1}(\omega) = \frac{1}{2\pi i} \int_{|z-z_0|=\epsilon} \frac{f'(z)}{f(z)-\omega} \cdot zdz.$$ 

10. Let $U := \{z : |z| < 2 \text{ and } |z-1| > \frac{1}{2} \text{ and } |z+1| > \frac{1}{2}\}$, and $f$ a holomorphic function on $U$. Recall that if $\gamma_1$ and $\gamma_2$ are closed chains in a region $\Omega$ in the complex plane, which are homologous in $\Omega$, then $\int_{\gamma_1} g(z)dz = \int_{\gamma_2} g(z)dz$, for every function $g$ holomorphic in $\Omega$, by the general form of Cauchy’s Theorem.

(a) Use Cauchy’s Theorem to prove that there exists a decomposition $f(z) = f_1(z) + f_2(z) + f_3(z)$, for all $z \in U$, where

$$f_1(z) = \sum_{n=1}^{\infty} \alpha_n (z-1)^{-n},$$

$$f_2(z) = \sum_{n=1}^{\infty} \beta_n (z+1)^{-n},$$

$$f_3(z) = \sum_{n=0}^{\infty} \gamma_n z^n,$$

and each of the series converges absolutely in $U$ and uniformly on compact subsets of $U$. Hint: Express $f(z)$ as an integral over three circles (indicate the domain $\Omega$ chosen in the application of Cauchy’s Theorem and provide a proof to any claim that two chains are homologous in $\Omega$).

(b) Prove that the above decomposition is unique. Hint: Relate $f_i$ to the Laurent series of $f$ in suitable annuli contained in $U$. 

2