Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1: Let $f: \mathbb{RP}^2 \to \mathbb{RP}^5$ be the map
\[ f([x_1, x_2, x_3]) = [x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3] \]
Prove that $f$ is an embedding.

Problem 2: Let $M$ be an $n$-dimensional smooth manifold and suppose $X_1, X_2$ are $C^\infty$-vector fields such that:
- $X_1(p), X_2(p)$ are linearly independent for all $p \in M$.
- $[X_1, X_2] = 0$.
Prove that for every $p \in M$ there exists a coordinate chart $(U; x_1, \ldots, x_n)$ such that $p \in U$ and, in $U$,
\[ X_i = \partial/\partial x_i ; \quad i = 1, 2. \]
Give an example to show that it may not be possible to take $U = M$.

Problem 3: Let $f: S^3 \to S^2$ be a smooth map. Let $\Omega_{S^2}$ denote the volume element of $S^2$ relative to the Euclidean metric.

a) Show that there exists a 1-form $\alpha$ in $S^3$ such that
\[ d\alpha = f^*(\Omega_{S^2}). \]

b) Show that
\[ I = \int_{S^3} \alpha \wedge d\alpha \]
is independent of the choice of $\alpha$.

c) Show that if $f$ is not surjective then $I = 0$.

Problem 4: Let $U(n) \subset GL_n(\mathbb{C})$ be the Lie group of all matrices satisfying $AA^* = I_n$, where the star denotes conjugate transpose.

a) Show that the diagonal subgroup of $U(n)$ is diffeomorphic to the compact torus $T^n$.

b) Explain why $\exp: M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ extends to a map $\exp: M_n(\mathbb{C}) \to GL_n(\mathbb{C})$. (Here $M_n(F)$ is the set of all $n \times n$ matrices with entries from $F$. You need not provide full details, but your answer should address the main points.)

c) Describe the Lie algebra $u(n)$ of $U(n)$ as a subspace of $M_n(\mathbb{C})$ (you may assume $M_n(\mathbb{C})$ is a Lie algebra).
Problem 5: Let \( M \) be an orientable, \( n \)-dimensional Riemannian manifold and \( N \subset M \) an \((n-1)\)-dimensional submanifold. Suppose there exists an open set \( U \) in \( M \) such that \( N \subset U \) and a vector field \( X \in \mathcal{X}(U) \) with the property that \( X(p) \not\in T_p(N) \) for all \( p \in N \).

a) Prove that \( N \) is orientable.

b) Express the volume element of \( N \) in terms of the vector field \( X \) and the volume element of \( M \), where \( N \) is given the metric induced from \( M \).

c) Apply to the case of \( M = \mathbb{R}^{n+1} \) (given the Euclidean metric), \( N = S^n \) to obtain an explicit formula for the volume element of \( S^n \).

Problem 6: Let \( M \) be an open subset of \( \mathbb{R}^2 \) with the usual orientation and the Riemannian metric
\[
ds^2 = (f(x,y))^2(dx^2 + dy^2),
\]
where \( f(x,y) \) is a smooth function which does not vanish anywhere on \( M \). Give explicit coordinate expressions for the following:

a) The volume element of \((M, ds^2)\).

b) \( \text{grad}(F) \), where \( F \) is a smooth function on \( M \).

c) \( \text{div}(X) \), where \( X \) is a smooth vector field on \( M \).

d) The Gaussian curvature of \((M, ds^2)\).

Problem 7: Let \((M, g)\) be a Riemannian manifold. We view \( g \) as an order 2 covariant tensor on \( M \) and define for \( X \in \mathcal{X}(M) \) the Lie derivative:
\[
L_Xg(Y, Z) := Xg(Y, Z) - g([X, Y], Z) - g(Y, [X, Z])
\]
a) Verify that \( L_Xg \) is a covariant tensor of order 2 on \( M \).

b) Suppose \( X \) is a complete vector field on \( M \) and let \( \theta^X_t \), \( t \in \mathbb{R} \), denote the corresponding one-parameter group of diffeomorphisms of \( M \). Prove that \( \theta^X_t \) is an isometry for all \( t \in \mathbb{R} \) if and only if \( L_Xg = 0 \).