University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 7 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

Problem 1: Let $f\colon \mathbb{RP}^2 \to \mathbb{RP}^5$ be the map

$$f([x_1, x_2, x_3]) = [x_1^2, x_2^2, x_3^2, x_1x_2, x_1x_3, x_2x_3]$$

Prove that f is an embedding.

Problem 2: Let M be an n-dimensional smooth manifold and suppose X_1, X_2 are C^∞-vector fields such that:

- $X_1(p), X_2(p)$ are linearly independent for all $p \in M$.
- $[X_1, X_2] = 0$.

Prove that for every $p \in M$ there exists a coordinate chart $(U; x_1, \dots, x_n)$ such that $p \in U$ and, in U,

$$X_i = \partial/\partial x_i \ ; \quad i = 1, 2.$$

Give an example to show that it may not be possible to take $U = M$.

Problem 3: Let $f\colon S^3 \to S^2$ be a smooth map. Let Ω_{S^2} denote the volume element of S^2 relative to the Euclidean metric.

a) Show that there exists a 1-form α in S^3 such that

$$d\alpha = f^*(\Omega_{S^2}).$$

b) Show that

$$I = \int_{S^3} \alpha \wedge d\alpha$$

is independent of the choice of α.

c) Show that if f is not surjective then $I = 0$.

Problem 4: Let $U(n) \subset GL_n(\mathbb{C})$ be the Lie group of all matrices satisfying $AA^* = I_n$, where the star denotes conjugate transpose.

a) Show that the diagonal subgroup of $U(n)$ is diffeomorphic to the compact torus T^n.

b) Explain why $\exp\colon M_n(\mathbb{R}) \to GL_n(\mathbb{R})$ extends to a map $\exp\colon M_n(\mathbb{C}) \to GL_n(\mathbb{C})$. (Here $M_n(F)$ is the set of all $n \times n$ matrices with entries from F. You need not provide full details, but your answer should address the main points.)

c) Describe the Lie algebra $u(n)$ of $U(n)$ as a subspace of $M_n(\mathbb{C})$ (you may assume $M_n(\mathbb{C})$ is a Lie algebra).

Problem 5: Let M be an orientable, n-dimensional Riemannian manifold and $N \subset M$ an $(n-1)$-dimensional submanifold. Suppose there exists an open set U in M such that $N \subset U$ and a vector field $X \in \mathcal{X}(U)$ with the property that $X(p) \notin T_p(N)$ for all $p \in N$.

a) Prove that N is orientable.
b) Express the volume element of N in terms of the vector field X and the volume element of M, where N is given the metric induced from M.
c) Apply to the case of $M = \mathbb{R}^{n+1}$ (given the Euclidean metric), $N = S^n$ to obtain an explicit formula for the volume element of S^n.

Problem 6: Let M be an open subset of $\mathbb{R}^2$ with the usual orientation and the Riemannian metric

$$ds^2 = (f(x,y))^2(dx^2 + dy^2),$$

where $f(x,y)$ is a smooth function which does not vanish anywhere on M. Give explicit coordinate expressions for the following:

a) The volume element of (M, ds^2).
b) $\mathrm{grad}(F)$, where F is a smooth function on M.
c) $\mathrm{div}(X)$, where X is a smooth vector field on M.
d) The Gaussian curvature of (M, ds^2).

Problem 7: Let (M,g) be a Riemannian manifold. We view g as an order 2 covariant tensor on M and define for $X \in \mathcal{X}(M)$ the Lie derivative:

$$L_X g(Y,Z) \ := \ Xg(Y,Z) - g([X,Y],Z) - g(Y,[X,Z])$$

a) Verify that $L_X g$ is a covariant tensor of order 2 on M.
b) Suppose X is a complete vector field on M and let θ_t^X, $t \in \mathbb{R}$, denote the corresponding one-parameter group of diffeomorphisms of M. Prove that θ_t^X is an isometry for all $t \in \mathbb{R}$ if and only if $L_X g = 0$.