Department of Mathematics and Statistics
University of Massachusetts

Basic Exam: Topology

January 25, 2008

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.
Passing standard: For Master's level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) (a) Let $f : X \to Y$ be surjective and continuous. Let $g : Y \to Z$ be a function such that $f \circ g$ is continuous. Show that if f is closed then g is continuous.
(b) Prove that f being closed is a necessary condition in part (a).

(2) (a) Show that if Y is compact then the projection $\pi\colon X \times Y \to X$ is a closed map.
(b) Does the result of (a) remain true if Y is not compact? Prove or give a counterexample.

(3) Let $\mathcal{C}$ be the space of all continuous functions defined on the interval $I = [0, 1]$. Define metrics on $\mathcal{C}$ by

$$d_1(f, g) = \mathrm{Sup}_{x \in I} |f(x) - g(x)| \quad \text{and} \quad d_2(f, g) = \int_I |f(x) - g(x)|\, dx.$$

Let $\mathcal{T}_1$, $\mathcal{T}_2$ be the topologies on $\mathcal{C}$ induced from these metrics.
(a) Prove that $\mathcal{T}_2$ is coarser than $\mathcal{T}_1$.
(b) Prove that $\mathcal{T}_1$ is not coarser than $\mathcal{T}_2$.

(4) Consider the real line equipped with a topology generated by the "half-open" intervals of the form

$$[a, b) \;=\; \{x \mid a \leq x < b\}.$$

This is called the *lower-limit* topology, and we denote it by $\mathbb{R}_l$. Let Q be the subspace of $\mathbb{R}_l^2$ defined by

$$Q = \{(q, -q) \mid q \in \mathbb{Q}\}.$$

Is Q open? Closed? Hausdorff? Connected? Justify.

(5) A topological space is called *locally compact* if for every point x there is a compact subspace containing a neighborhood of x.

   (a) Show that any set endowed with the discrete topology becomes a locally compact topological space.

   (b) Show that the rational numbers, with the subspace topology inherited from the real numbers, is not locally compact.

   (c) Give an example of a locally compact space X and a continuous map $f\colon X \to Y$ such that the image $f(X)$ is not locally compact.

   (d) Assume X is locally compact and $f\colon X \to Y$ is continuous and open. Show that $f(X)$ is locally compact.

(6) A family of sets F in X is said to be *locally finite* if for any point $x \in X$ and any open neighborhood U of x, there are at most a finite number of elements of F that intersect U non-trivially.

   (a) Show that if F is a locally finite family of closed sets then the set

$$\bigcup_{U \in F} U$$

   is closed.

   (b) Show that if F is a locally finite closed cover of X and $f\colon X \to Y$ is a function with the property that for each $U \in F$, the restriction $f|_U$ is continuous, then f is continuous.

(7) Let X, Y be spaces, and take a surjective map $p\colon X \to Y$, not necessarily continuous.

Assume that for *any* topological space Z and any function $g\colon Y \to Z$ we have:

$$g \text{ is continuous} \iff g \circ p\colon X \to Z \text{ is continuous.}$$

Show that Y has the quotient topology induced by p.

(Hint: you can start by considering the case where $Z = Y$ *as sets*, where Z has the quotient topology, and g is the identity map.)

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