Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master’s level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

**Notation.** For \( w \in \mathbb{C} \) and \( r \in \mathbb{R}_{>0} \), we let \( B_r(w) = \{ z \in \mathbb{C} : |z - w| < r \} \) be the open disc of radius \( r \) centered at \( w \). We let \( D = B_1(0) \) be the open unit disc.

1. Suppose \( f \) is an entire function and \( M, R \) are positive constants such that \( |f(z)| \leq M \) on the circle \( |z| = R \). Determine, with proof, a quantity \( N \) which depends only on \( k \) and \( M \) (not on \( f, r, R \)) such that for all \( 0 \leq r < R \),
\[
|f^{(k)}(re^{i\theta})| \leq \frac{N}{(R-r)^k}, \quad k = 0, 1, 2, \ldots
\]

2. Suppose the coefficients of the power series \( f(z) = \sum_{n \geq 0} a_n z^n \) satisfy the recurrence relation
\[
a_0 = a_1 = 1; \quad a_n - 7a_{n-1} + 12a_{n-2} = 0, \quad n = 2, 3, 4, \ldots
\]
Determine the function \( f \) explicitly as well as the radius of convergence of the given power series.

3. Prove that for a fixed complex number \( w \in \mathbb{C} \),
\[
\frac{1}{2\pi} \int_0^{2\pi} e^{2w \cos(\theta)} \, d\theta = \sum_{n=0}^{\infty} \left( \frac{w^n}{n!} \right)^2.
\]

4. Let \( s(y) \) and \( t(y) \) be real differentiable functions of \( y \) on \( -\infty < y < \infty \) satisfying \( s(0) = 1, t(0) = 0 \), with the property that the complex function
\[
f(x + iy) = e^x(s(y) + it(y))
\]
is entire. Determine \( s(y) \) and \( t(y) \) (with proof).

5. Prove or disprove.
   (1) The image of \( \mathbb{C} \) under a non-constant entire function is dense in \( \mathbb{C} \).
   (2) If the radius of convergence of the series \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) is \( R \), then the radius of convergence of the series \( \sum_{n=1}^{\infty} \frac{a_n}{n!}(z - z_0)^n \) is also \( R \).
   (3) If \( f(z) \) is analytic in a domain \( D \) and and \( u(x, y) = Re(f(x + iy)) \), then the function \( H(x, y) = \sin(u(x, y)) \) is harmonic in \( D \).
6. (a) State, but do not prove, the Schwartz Lemma.
   (b) Let \( f \) be a holomorphic function from the open unit disc \( D \) into itself. Prove the inequality
   \[
   \frac{|f'(z)|}{1 - |f(z)|^2} \leq \frac{1}{1 - |z|^2},
   \]
   for all \( z \in D \).

7. Using a contour integral, compute
   \[
   \int_0^\infty \frac{\cos(x) - 1}{x^2} \, dx.
   \]
   Justify all estimates.

8. Let \( f \) be a one-to-one holomorphic map from an (open, connected) region \( D_1 \) onto a region \( D_2 \). Suppose that \( D_1 \) contains the closure of the unit disk \( \mathbb{D} = \{ z : |z| < 1 \} \). Prove that for \( w \in f(\mathbb{D}) \), the inverse function \( f^{-1}(w) \) is given by
   \[
   f^{-1}(w) = \frac{1}{2\pi i} \oint_{|z|=1} \frac{f'(z)}{f(z) - w} \, dz,
   \]
   where the circle \( |z| = 1 \) is traversed once counterclockwise.

9. (a) Compute
   \[
   \int_C \frac{dz}{(z - i)^2 \cos(z)},
   \]
   where \( C \) denotes the circle \( \{ z : |z| = 4 \} \) traversed once counterclockwise.
   (b) Let \( C \) be the circle \( \{ z : |z| = 2 \} \) traversed once counterclockwise. Compute the integral
   \[
   \int_C \frac{z^2(5z - 1)^8}{1 - z^{10}} \, dz.
   \]

10. Suppose \( f \) is holomorphic in a connected and simply connected open region \( \Omega \) and satisfies \( |f(z) - 1| < 1 \) for all \( z \in \Omega \). Suppose \( C \) is a closed contour in \( \Omega \) traversed once counterclockwise. Determine, with proof, the value of the integral
    \[
    \oint_C \frac{f'(z)}{f(z)} \, dz.
    \]