DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. An electric circuit is described by the system of ordinary differential equations

$$\begin{aligned} \frac{dI}{dt} &= \frac{1}{L}V \\ \frac{dV}{dt} &= -\frac{1}{C}I - \frac{1}{RC}V. \end{aligned}$$

   where I is the current through the inductor and V is the voltage drop across the capacitor.

   (a) Show that the eigenvalues of the coefficient matrix are real and different if $L > 4R^2C$; show that they are complex conjugates if $L < 4R^2C$.

   (b) Suppose that $R = 1$ ohm, $C = \frac{1}{2}$ farad, and $L = 1$ henry. Find the general solution of the system in this case.

   (c) Find $I(t)$ and $V(t)$ if $I(0) = 2$ amperes and $V(0) = 1$ volt.

   (d) For the circuit of part (b) determine the limiting values of $I(t)$ and $V(t)$ as $t \to \infty$. Do these limiting values depend on the initial conditions?

2. (a) Solve using the method of characteristics the following equation

$$u_t + (x+1)u_x = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$

   with initial datum $u(x, 0) = f(x)$.

   (b) Can you solve using the method of characteristics,

$$u_t + x^2u_x = 0\,, \quad -\infty < x < \infty\,, \quad t > 0\,,$$

   and initial datum $u(x, 0) = f(x)$? Explain the limitations, if any.

3. Consider the wave equation with a transverse elastic force

$$\rho u_{tt} - T u_{xx} + ku = 0\,, \quad -\infty < x < \infty, \quad t > 0\,,$$

with initial data

$$u(x,0) = \phi(x)\,, \quad u_t(x,0) = \psi(x)$$

and assuming that u and its derivatives vanish at $\pm\infty$. Here ρ, T, and k are positive constants.

(a) Show that the total energy

$$E(t) = \frac{1}{2}\int_{\infty}^{\infty} \rho u_t^2(x,t) + T u_x^2(x,t) + k u^2(x,t) dx\,,$$

is constant in time.

(b) Using (a), show that the solution to the initial value problem is unique.

4. Consider the equation

$$u_t - k u_{xx} + au = 0\,, \quad 0 < x < l, \quad t > 0\,,$$

with initial data

$$u(x,0) = \phi(x)\,,$$

and boundary conditions

$$u(0,t) = 0 = u(l,t)\,,$$

where k and a are constants and k is positive.

(a) Write the series solution for this problem.

(b) Can you conjecture from the formula obtained in (a) the behavior of the solution $u = u(x,t)$ for large values of t?

5. Consider the initial value problem

$$x'' + (1+\epsilon)x = 0\,, \quad x(0) = 1\,, x'(0) = 0\,. \tag{2}$$

where ϵ is positive and small.

(a) Using the regular perturbation method find the first 2 terms, x_0, x_1 in the series expansion.

(b) Find the exact solution to equation 2.

(c) If there is disagreement, suggest an alternative series expansion and calculate the first term; how does it compare to the exact solution?

6. Consider the equation

$$x'' + x + \epsilon x^3 = 0$$

(a) Show that it corresponds to a conservative system and calculate the conserved quantity.

(b) Show that if $\epsilon > 0$, $(0, 0)$ is a nonlinear center.

(c) How do the trajectories close to $(0, 0)$ behave if $\epsilon < 0$?