• Do 7 of the following 9 problems.
• **Passing Standard:**
  • For Master’s level: 60% with three questions essentially complete (including at least one from each part)
  • For Ph. D. level: 75% with two questions from each part essentially complete.
• Show your work!

**Part I. Linear Algebra**

1. Denote by $D = \frac{d}{dx}$ the differential operator on the set $\mathcal{P}$ of all real, one-variable polynomials of degree $\leq 3$ (including the zero polynomial). It is a fact that $\mathcal{P}$ is a real vector space and that $D$ is a linear transformation from $\mathcal{P}$ to itself. Determine the characteristic polynomial and the minimal polynomial of this operator $D$.

2. Let $A$ be a complex $n \times n$ matrix for which $A^3 = A$. Prove that rank$(A) = \text{trace}(A^2)$.

3. We say that a real, $n \times n$ symmetric matrix $A$ is **positive definite** if $(A\vec{x}) \cdot \vec{x} > 0$ for all non-zero $\vec{x} \in \mathbb{R}^n$, where $\cdot$ denotes the usual inner product on $\mathbb{R}^n$.

   (a) Show that $A$ is positive definite if and only if all of its eigenvalues are positive.
   
   (b) If $A$ is positive definite, show that there exists another positive definite matrix $B$ such that $A = B^2$.

4. For a vector subspace $W$ of $\mathbb{R}^n$, the orthogonal complement of $W$ is defined by

   $$W^\perp := \{ \vec{x} \in \mathbb{R}^n : \vec{x} \cdot \vec{w} = 0 \text{ for all } \vec{w} \in W \},$$

   where $\cdot$ denotes the usual inner product on $\mathbb{R}^n$. Show that $(W^\perp)^\perp = W$ for every subspace $W$ of $\mathbb{R}^n$. 

Part II. Advanced Calculus

1. Prove directly the following special case of the Arithmetic-Geometric Mean inequality: For any integer $n \geq 1$, if $y_1, \ldots, y_n$ are positive real numbers with product 1, then $y_1 + \cdots + y_n \geq n$.

2. For each integer $n \geq 1$, let $f_n(x) = n^2 x^n (1 - x)$.
   (a) Show that this sequence of functions converges pointwise on $[0, 1]$.
   (b) Does this sequence of functions converges uniformly on $[0, 1]$?
   (c) Does $\lim_{n \to \infty} \int_0^1 f_n(x) \, dx = \int_0^1 \left( \lim_{n \to \infty} f_n(x) \right) \, dx$?

3. Let $f(x)$ be a function which is continuously differentiable on the closed interval $[a, b]$. If $f(x)$ is not linear, show that there exists a number $c \in (a, b)$ at which
   $$|f'(c)| > \left| \frac{f(b) - f(a)}{b - a} \right|.$$
   (note: this is not the mean-value theorem!)

4. Calculate
   $$\int \int_S (\nabla \times \vec{F}) \cdot \vec{n} \, dS,$$
   where $S$ is the surface
   $$S = \{(x, y, z) : x^2 + y^2 = 1, -1 \leq z \leq 0\} \cup \{(x, y, z) : x^2 + y^2 \leq 1, z = -1\},$$
   oriented by its outward-point normal vector $\vec{n}$, and
   $$\vec{F}(x + \vec{i} + y\vec{j} + z\vec{k}) = (y + e^{xz})\vec{i} - (x + e^{yz})\vec{j} + (e^{yz})\vec{k}.$$

5. Suppose $f(x)$ is Riemann-integrable on $[0, 1]$ with $0 < f(x) < 1$. Show that $\int_0^1 f(x) \, dx > 0$. 