University of Massachusetts  
Department of Mathematics and Statistics  
Advanced Exam in Geometry  
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Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

1. Let $F, G : \mathbb{R}^{n+1} \to \mathbb{R}$ be $C^\infty$ maps and suppose that $0 \in \mathbb{R}$ is a regular value of $F$. Set $X := F^{-1}(0)$ and $g := G|_X : X \to \mathbb{R}$. Prove that a point $p \in X$ is a critical point of $g$ if and only if there exists $\lambda \in \mathbb{R}$ such that

$$dF_p = \lambda dG_p.$$  

2. Let $M$ be a 3-dimensional smooth manifold and let $\alpha$ be a nowhere zero smooth 1-form on $M$. Define: $\Delta(p) := \{ v \in T_p(M) \mid \alpha(p)(v) = 0 \}$.

(a) Prove that $\Delta$ is a smooth 2-dimensional distribution in $M$.

(b) Prove that $\Delta$ is integrable (involutive) if and only if $\alpha \wedge d\alpha = 0$.

3. Prove or disprove the following statements:

(a) If a 1-form $\alpha$ on a smooth manifold $M$ is nowhere zero and $\theta$ is another 1-form such that $\theta \wedge \alpha = 0$, then there exists $f \in C^\infty(M)$ such that $\theta = f \alpha$.

(b) If a 1-form $\alpha$ on $M = \mathbb{R}^2\setminus\{0\}$ satisfies $d\alpha = 0$, then there exists $f \in C^\infty(\mathbb{R}^2\setminus\{0\})$ such that $\alpha = df$.

4. Let $M$ be an $n$-dimensional smooth manifold, and let $S \subset M$ be an embedded submanifold of dimension $n-1$. Show that the normal bundle $N_S M := (TM|_S)/TS$ is trivial if and only if there exists an open neighborhood $U \subset M$ of $S$ and a smooth function $f \in C^\infty(U)$ so that $0$ is a regular value of $f$ and $S = f^{-1}(0)$.

5. A manifold $M$ is called symplectic if there exists a 2-form $\omega \in \Lambda^2(M)$ which satisfies (1) $d\omega = 0$ and (2) $\omega_p$ is a nondegenerate bilinear form on $T_pM$ for all $p \in M$. Prove that

(a) any orientable, 2-dimensional manifold is symplectic, and

(b) For any $n \geq 2$, the sphere $S^{2n}$ is not symplectic but the torus $T^{2n} = (S^1)^{2n}$ is.
6. Let \((M, g)\) be a connected, orientable Riemannian manifold.

(a) Define the divergence, \(\text{div}_g(X)\), of a vector field \(X\) on \(M\).
(b) Prove that \(\text{div}_g(X)\) is independent of the choice of orientation.
(c) Let \(\lambda \in C^\infty(M)\) and let \(g_\lambda\) the metric, conformal to \(g\), defined by

\[
g_\lambda(X, Y) = e^{2\lambda} g(X, Y).
\]

Prove that

\[
\text{div}_{g_\lambda}(X) = \text{div}_g(X) + (\dim M) X(\lambda).
\]

7. Let \(M = \{(x, y) \in \mathbb{R}^2 : y > 0\}\), with the metric \(g = y \, dx^2 + dy^2\), i.e.

\[
g(\partial/\partial x, \partial/\partial x) = y \; ; \; g(\partial/\partial y, \partial/\partial y) = 1 \; ; \; g(\partial/\partial x, \partial/\partial y) = 0.
\]

(a) Compute the Gaussian curvature of \((M, g)\).
(b) Given that the Christoffel symbols of \(g\) relative to the frame \(\partial/\partial x, \partial/\partial y\) are given by:

\[
\Gamma^1_{11} = \Gamma^2_{12} = \Gamma^1_{22} = \Gamma^2_{22} = 0 \; ; \; \Gamma^1_{11} = -1/2 \; ; \; \Gamma^1_{12} = 1/(2y),
\]

write the differential equations for a geodesic in \((M, g)\).
(c) Determine whether vertical or horizontal lines are geodesics and, if so, what is the appropriate parametrization.

8. Suppose that a Lie group \(G\) acts on a smooth manifold \(M\); i.e., there is a smooth map \(a: G \times M \to M\) satisfying

\[
a(g, a(h, p)) = a(gh, p)
\]

for all \(g, h \in G\) and \(p \in M\).

(a) Show that for any \(p \in M\), the stabilizer

\[
G_p = \{g \in G \mid a(g, p) = p\}
\]

is a Lie subgroup of \(G\).
(b) Show that there is a unique linear transformation \(X \mapsto \tilde{X}\) from the Lie algebra \(\text{Lie} G\) to the space of smooth vector fields on \(M\) such that

\[
\tilde{X}_{a(g, p)} = (\phi_p)_*(X_g)
\]

for all \(p \in M, g \in G\), where \(\phi_p: G \to M\) is given by \(\phi_p(g) = a(g, p)\).
(c) Show that \(\text{Lie} G_p = \{X \in \text{Lie} G \mid \tilde{X}_p = 0\}\)