Do eight out of the following 10 questions. Each question is worth 10 points. To pass at the Master’s level it is sufficient to have 45 points, with 3 questions essentially correct; 55 points with 4 questions essentially correct are sufficient for passing at the Ph.D. level.

Note: All answers should be justified.

1. Prove that there does not exist a one-to-one conformal map from the punctured unit disc \( \{ z : 0 < |z| < 1 \} \) onto the annulus \( A = \{ z : 1 < |z| < 2 \} \).

2. Find a one-to-one conformal map from the region \( \{ z : 1 < |z| < 2, \text{ and } \Re(z) > 0 \} \) onto the rectangle \( \{ x + iy : 0 < x < \pi \text{ and } 0 < y < \ln(2) \} \).

3. State and prove the Swartz Lemma.

4. (a) Find the Laurent series expansion of the function \( f(z) = \frac{1}{z^2 - 4z + 3} \) valid near and centered at \( z_0 = 1 \). For what values of \( z \) does the series converges?
   (b) Find the radius of convergence \( R \) of the Taylor series about \( z = 1 \) of the function \( f(z) = \frac{1}{1 + z^2 + z^4 + z^6 + z^8 + z^{10}} \).

   Express the answer explicitly as a real number.

5. Let \( f(z) \) be an analytic function on the punctured complex plane \( \mathbb{C} \setminus \{ 0 \} \), satisfying
   \[
   |f(z)| \geq \frac{1}{|z|^d},
   \]
   for some real number \( d \). Show that \( d \) must be an integer and there exists a constant \( c \in \mathbb{C} \), such that \( f(z) = cz^{-d} \).

   Hint: Reduce to the case \( 0 < d \leq 1 \) and analyze the singularities of \( f \).

6. Prove that every one-to-one holomorphic map \( f \) from the upper-half-plane \( \mathbb{H} := \{ x + iy : x, y \in \mathbb{R}, y > 0 \} \) onto itself is a fractional linear transformation with real coefficients and positive determinant. That is, \( f \) can be written in the form:
   \[
   f(z) = \frac{az + b}{cz + d},
   \]
   where \( a, b, c, d \in \mathbb{R} \), and \( ad - bc = 1 \).

7. (a) Prove that the series
   \[
   \sum_{n=-\infty}^{\infty} \frac{1}{(z - n)^2}
   \]
   defines a meromorphic function \( f(z) \), periodic with period 1, over the complex plane.
(b) Prove that the function \( g(z) := f(z) - \frac{\pi^2}{\sin^2(\pi z)} \) is an entire function.

8. Evaluate the following integrals

(a) \( \int_C \frac{\cos(z)dz}{z^2(z^5 - 1)} \), where \( C \) is the circle \( \{|z| = \frac{1}{2}\} \).

(b) \( \int_C \frac{z^4 \cos(1/z)}{z^5 + 1} dz \) where \( C \) is the circle \( \{|z| = 3\} \).

9. Evaluate the integral \( \int_0^\infty \frac{\cos(x)dx}{x^2 + 4} \). Justify all your steps!!!

10. Let \( f \) be a non-constant entire function and \( C := \{z : |z| = 1\} \) the unit circle. Suppose \( |f(z)| = 1 \), for all \( z \in C \). Prove that the winding number \( W(f(C), 0) := \frac{1}{2\pi i} \int_C \frac{f'(z)dz}{f(z)} \) is positive.