DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS
MASTER'S OPTION EXAM-APPLIED MATHEMATICS
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Do five of the following problems. All problems carry equal weight.
Passing level: 60% with at least two substantially correct.

1. Consider the system

$$\begin{aligned}\frac{dx}{dt} &= 1 - xy \\ \frac{dy}{dt} &= x - y^3.\end{aligned}$$

   (a) Determine all real critical points of the system.

   (b) Find the corresponding linear system near each critical point.

   (c) Discuss the stability of the solution near each critical point.

2. A particle of mass m is moving along the $x-$axis, subject to a nonlinear force $F(x)$. The equation of motion is

$$m\ddot{x} = F(x)$$

   Let $F(x) = -\frac{dV}{dx}$ where $V(x)$ denotes the potential energy. The kinetic energy is defined to be $\frac{1}{2}m\dot{x}^2$. Show that the kinetic plus the potential energy is conserved in time.

3. Two species of fish that complete with each other for food, but do not prey on each other are the bluegill and redear, Suppose that the pond is stocked with bluegill and redear and let x and y be the populations of bluegill and redear respectively, at time t. Suppose that the

competition is modeled by the ODEs

$$\frac{dx}{dt} = x(a_1 - b_1 x - c_1 y)$$

$$\frac{dy}{dt} = y(a_2 - b_2 y - c_2 x)$$

If $a_2/c_2 > a_1/b_1$ and $a_2/b_2 > a_1/c_1$, show that the only possible equilibrium populations in the pond have either (1) no fish or (2) no bluegill or (3) no redear. What happens as time approaches infinity?

4. Solve the diffusion equation $u_t = ku_{xx}$ in $0 < x < L$, with the mixed boundary conditions $u(0,t) = u_x(L,t) = 0$ and initial conditions $u(x,0) = \phi(x)$.

5. Consider the interval $0 \leq x \leq 1$ of length one and consider the eigenvalue problem

$$-X'' = \lambda X$$

$$X'(0) + X(0) = 0 \text{ and } X(1) = 0$$

(absorption at one end and zero at the other),

(a) Find an eigenfunction with eigenvalue zero.

(b) Find an equation for the positive eigenvalues $\lambda = \beta^2$.

(c) Show graphically from part (b) that there are an infinite number of positive eigenvalues.

6. Consider the partial differential equation

$$6u_x + xu_y = y$$

Given that $u(3,3) = 4$, what value of $u(15,21)$ is obtained by following the characteristic curve?

7. Use the Fourier method to find the solution of the boundary value problem

$$-\triangle u + a^2 u = 0 \text{ in } 0 < x < 1, 0 < y < 1$$

$$u(x,0) = \phi(x), u(x,1) = 0, u(0,y) = 0, u(1,y) = 0$$

where ϕ is smooth enough, a is real and $\phi(0) = \phi(1) = 0$.