Do five of the following problems. All problems carry equal weight.
Passing level: 75% with at least three substantially complete solutions.

1a.) Show that
\[ E(x, y) = \frac{1}{2} y^2 - \cos x \]
is non-increasing on all solutions \((x(t), y(t))\) of
\[ \begin{align*}
  x' &= y \\
  y' &= -cy - \sin x,
\end{align*} \]
for any given constant \(c \geq 0\).

1b.) Describe the structure of the global phase plane of (*) in the region
\[ \{(x, y) : -\frac{3\pi}{2} < x < \frac{3\pi}{2}\} \]
when
\[ \begin{align*}
  (1) & \quad c = 0, \\
  (2) & \quad c = 3.
\end{align*} \]
Complete answers should include sketches depicting all periodic and/or connecting orbits (if any), and the local behavior of all solutions near rest points (in each case), and should be supported by accompanying analytical calculations and arguments.

2a.) Suppose that \(f : \mathbb{R}^n \to \mathbb{R}^n\) and \(g : \mathbb{R}^n \to \mathbb{R}\) are smooth and that a bounded domain \(\sigma \subset \mathbb{R}^n\) is defined by \(\sigma = \{x : g(x) < 0\}\). Suppose that there is a constant \(\delta > 0\) with \(\nabla g(x) \cdot f(x) < -\delta\) for all \(x \in \partial \sigma\). Prove that if \(x(t)\) is the solution to \(x' = f(x), x(0) = x_0\), then \(x_0 \in \sigma\) implies that \(x(t) \in \sigma\) for all \(t \geq 0\).
2b.) Consider the nonautonomous initial value problem
\[
\frac{dx}{dt} = f(x, t), \quad x(0) = x_0.
\]
Suppose that a tube-like region \( \Sigma \) in \( \mathbb{R}^n \times \mathbb{R} \) is defined as the union (over \( t \in \mathbb{R} \)) of bounded domains
\[
\sigma_t = \{ x \in \mathbb{R}^n : G(x, t) < 0 \}
\]
for some smooth function \( G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R} \). Formulate a condition under which \( (x_0, 0) \in \Sigma \) implies that \( (x(t), t) \in \Sigma \) for all \( t \geq 0 \).

3.) Let \( \Omega \subseteq \mathbb{R}^n \) be a smooth, bounded domain, and for every \( \tau \geq 0 \) let \( w = w(x, t; \tau) \) denote the solution of the initial-value problem:
\[
\begin{cases}
    w_t - \Delta w = 0 & x \in \Omega, \quad t > \tau \\
    w = 0 & x \in \partial \Omega, \quad t > \tau \\
    w|_{t=\tau} = f(x, \tau) & x \in \Omega,
\end{cases}
\]
where \( f(x, t) \) is a given function defined and smooth for \( x \in \Omega, t \geq 0 \). Express the solution \( u = u(x, t) \) of the problem
\[
\begin{cases}
    u_t - \Delta u = f(x, t) & x \in \Omega, \quad t > 0 \\
    u = 0 & x \in \partial \Omega, \quad t > 0 \\
    u|_{t=0} = 0 & x \in \Omega,
\end{cases}
\]
in terms of \( w \) and fully justify this expression.

*HINT:* To infer the required expression consider the analogous ODE system
\[
\dot{x} = Ax + f(t) \quad \text{with} \ x(0) = 0.
\]
4.) Consider the equation for a vibrating string with “internal damping” (involving the rather unusual $u_{xxt}$ term):

\[
\begin{align*}
(\star) & \quad \left\{ \begin{array}{l}
\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} + \epsilon^2 \dfrac{\partial^4 u}{\partial x^4} \\
u(0, t) = 0 = u(1, t)
\end{array} \right. \\
in \quad 0 < x < 1, \ t > 0
\end{align*}
\]

(i) Show that any solution of (\star) satisfies

\[
\frac{dE}{dt} \leq 0 \quad \text{with} \quad E(t) = \frac{1}{2} \int_0^1 (u_t^2 + u_x^2) \, dx.
\]

(ii) Use the result (i) to deduce the uniqueness of the solution of the initial value problem for (\star).

5.) Let $\Omega_a = \{(x, y) : 0 < x < a, 0 < y < 1\}$.

(i) Find the smallest number $a > 0$ such that the problem

\[
\begin{align*}
(\star) & \quad \left\{ \begin{array}{l}
\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + 13u = f \\
u = 0 \quad \text{on} \quad \partial \Omega_a
\end{array} \right. \\
in \quad \Omega_a
\end{align*}
\]

can have more than one solution for some function $f = f(x, y)$.

(ii) For the value of $a$ found in part (i) discuss solving (\star) when $f(x, y) = \sin \pi y$.

6.) State and prove the classical maximum principle for the initial boundary value problem for the heat equation:

\[
\begin{align*}
\left\{ \begin{array}{l}
\dfrac{\partial u}{\partial t} - \Delta u = 0 \\
u(x, t) = \varphi(x, t) \\
u(x, 0) = u_0(x)
\end{array} \right. \\
\text{for} \quad (x, t) \in \Omega \times (0, T) \\
\text{for} \quad x \in \partial \Omega, \ 0 \leq t \leq T \\
\text{for} \quad x \in \bar{\Omega}
\end{align*}
\]

for general (regular) boundary data $\varphi$ and initial data $u_0$, with $\varphi(x, 0) = u_0(x)$. 