University of Massachusetts  
Dept. of Mathematics and Statistics  
Basic Exam - Topology  
January 27, 2006

Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing Standard: For Master’s level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

1. Let $X, Y$ be topological spaces with $X$ compact. Let $b \in Y$ and let $U \subset X \times Y$ be an open subset which contains $X \times \{b\}$. Show that there exists a neighborhood $V$ of $b$ such that $X \times V \subset U$.

2. Let $A \subset \mathbb{R}$ be compact, and let $B \subset \mathbb{R}$ be closed.
   (a) Show that the set $C = \{a + b \mid a \in A, b \in B\}$ is closed.
   (b) Find an example of closed sets $A$ and $B$ in $\mathbb{R}$ for which $C \subset \mathbb{R}^2$ is not closed.

3. Let $A$ and $B$ be proper subsets of spaces $X$ and $Y$, respectively. If $X$ and $Y$ are both connected, show that $(X \times Y) - (A \times B)$ is connected. (Hint: Try looking at the case $X = Y = [0, 1], A = B = (0, 1)$).

4. Let $X = (0, 1], Z = [0, 1)$ and $Y = [0, 1]$. Let $f_n : Y \to Y$ be the continuous function whose graph consists of segments from $(0, 1)$ to $(\frac{1}{n}, 0)$ and from $(\frac{1}{n}, 0)$ to $(0, 1)$. Let $g_n : X \to Y$ be the restriction of $f_n$ to $(0, 1] = X$. Let $h_n : Z \to Y$ be the restriction of $f_n$ to $[0, 1) = Z$.
   (a) Does the sequence $h_n$ converge for the uniform metric $D$ on the set $C(X, Y)$ of continuous functions from $X$ to $Y$. (Here, $D(f, g) = \sup_{x \in X} |g(x) - f(x)|$).
   (b) Does the sequence $g_n$ converge pointwise?
   (c) Does the sequence $g_n$ converge in the compact open topology on $C(X, Y)$?
   (d) Does the sequence $h_n$ converge in the compact open topology on $C(Z, Y)$?

5. Let $(M, d)$ be a metric space and suppose $K$ and $H$ are subsets of $M$. For $x \in M$, define $d(x, k) = \inf_{y \in K} d(x, y)$ and define $d(H, K) = \inf_{x \in H} d(x, K)$.
   (a) Prove that if $K$ is closed and $H$ is compact, then $d(H, K) = 0$ if and only if $H \cap K \neq \emptyset$. 

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(b) Show by the way of an example that if $K$ and $H$ are closed in $M$, then it is possible for $H \cap K = \emptyset$ and $d(H, K) = 0$. (Hint: Find an example where $M = \mathbb{R}^2$.)

6. Let $X = \mathbb{R}^n / \sim$ be the quotient of $\mathbb{R}^n$ by the equivalence relation: $x \sim y$ if the difference vector $x - y$ has integer coordinates. Show that:

(a) $X$ is connected.
(b) $X$ is compact.
(c) $X$ is Hausdorff.

7. Let $X$ be a metric space.

(a) Show that $X$ has a countable dense subset if and only if $X$ has a countable base for its topology.
(b) Suppose that $X$ is compact. Show that both conditions from part (a) hold (you only need to show one!).