Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master’s level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

1. Use contour integration to evaluate the integral
   \[ \int_{-\infty}^{\infty} \frac{dx}{(1 + x^2)^2}. \]
   Justify all steps.

2. Let \( n \) be a positive integer, and \( 0 < \alpha < \pi \). Prove that
   \[ \frac{1}{2\pi i} \int_C \frac{z^n}{1 - 2z \cos \alpha + z^2} \, dz = \frac{\sin n\alpha}{\sin \alpha}, \]
   where \( C \) is the circle \(|z| = 2\) traversed once counterclockwise.

3. Suppose \( f \) is an entire function with the property that \( f(z)/z \to 0 \) as \(|z| \to \infty\). Show that \( f \) is constant.

4. Show that there is no function \( f \) analytic on the punctured plane \( \mathbb{C} - \{0\} \) that satisfies
   \[ |f(z)| \geq \frac{1}{\sqrt{|z|}} \quad \forall z \neq 0. \]

5. Let \( f(z) = a_0 + a_1 z + \cdots + a_n z^n \) be a polynomial of degree \( n > 0 \). Prove that
   \[ \frac{1}{2\pi i} \int_C z^{n-1}|f(z)|^2 \, dz = a_0 \overline{a_n} R^{2n}, \]
   where \( C \) is the circle \(|z| = R\) traversed once counterclockwise.
6. Prove that an injective entire function is necessarily linear. In other words, if $f$ is entire and $f(z) \neq f(w)$ whenever $z \neq w$, then there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $f(z) = az + b$ for all $z$. [Hint: consider $f(1/z)$].

7. (a) How many zeros (counting multiplicities) does the function
$$f(z) = 5z^{10} - e^z$$
have inside the unit disc?
(b) Are these all simple zeros of $f$? Explain.

8. Suppose $f : H \to \mathbb{C}$ is analytic on the right-half-plane
$$H = \{ z \in \mathbb{C} \mid \text{Re}(z) > 0 \}$$
and suppose $|f(z)| \leq 1$ for all $z \in H$. Furthermore, suppose $f(1) = 0$. What is the largest possible value for $|f'(1)|$? (Justify all steps).

9. Let $f(z) = \pi \cot(\pi z)$ have the Laurent expansion
$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$$
on the annulus $1 < |z| < 2$. Compute the the coefficients $a_n$ for $n < 0$. [Hint: express the coefficients in terms of integrals and use the Residue formula].

10. Prove the Argument Principle: Suppose $f$ is meromorphic in an open set $\Omega$ containing a circle $C$ as well as the interior of $C$. If $f$ has no zeros or poles on $C$, then
$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = Z - P$$
where $Z$ is the number of zeros of $f$ inside $C$ and $P$ is the number of poles of $f$ inside $C$ (counted with their multiplicities).