BASIC EXAM – COMPLEX ANALYSIS

25 JANUARY 2006

Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master's level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

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1. Use contour integration to evaluate the integral

$$\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}.$$

Justify all steps.

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2. Let n be a positive integer, and $0 < \alpha < \pi$. Prove that

$$\frac{1}{2\pi i}\int_C \frac{z^n}{1-2z\cos\alpha + z^2}dz = \frac{\sin n\alpha}{\sin\alpha},$$

where C is the circle $|z| = 2$ traversed once counterclockwise.

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3. Suppose f is an entire function with the property that $f(z)/z \to 0$ as $|z| \to \infty$. Show that f is constant.

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4. Show that there is no function f analytic on the punctured plane $\mathbb{C} - \{0\}$ that satisfies

$$|f(z)| \geq \frac{1}{\sqrt{|z|}} \qquad \forall z \neq 0.$$

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5. Let $f(z) = a_0 + a_1 z + \cdots + a_n z^n$ be a polynomial of degree $n > 0$. Prove that

$$\frac{1}{2\pi i}\int_C z^{n-1}|f(z)|^2 dz = a_0\overline{a_n}R^{2n},$$

where C is the circle $|z| = R$ traversed once counterclockwise.

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6. Prove that an injective entire function is necessarily linear. In other words, if f is entire and $f(z) \neq f(w)$ whenever $z \neq w$, then there exist $a, b \in \mathbb{C}$ with $a \neq 0$ such that $f(z) = az + b$ for all z. [Hint: consider $f(1/z)$].

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7. (a) How many zeros (counting multiplicities) does the function

$$f(z) = 5z^{10} - e^z$$

have inside the unit disc?

(b) Are these all simple zeros of f? Explain.

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8. Suppose $f : H \to \mathbb{C}$ is analytic on the right-half-plane

$$H = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$$

and suppose $|f(z)| \leq 1$ for all $z \in H$. Furthermore, suppose $f(1) = 0$. What is the largest possible value for $|f'(1)|$? (Justify all steps).

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9. Let $f(z) = \pi \cot(\pi z)$ have the Laurent expansion

$$\pi \cot(\pi z) = \sum_{n=-\infty}^{\infty} a_n z^n$$

on the annulus $1 < |z| < 2$. Compute the the coefficients a_n for $n < 0$. [Hint: express the coefficients in terms of integrals and use the Residue formula].

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10. Prove the Argument Principle: Suppose f is meromorphic in an open set Ω containing a circle C as well as the interior of C. If f has no zeros or poles on C, then

$$\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = Z - P$$

where Z is the number of zeros of f inside C and P is the number of poles of f inside C (counted with their multiplicities).
