University of Massachusetts
Department of Mathematics and Statistics
Advanced Exam in Geometry
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Do 5 out of the following 8 problems. Indicate clearly which questions you want graded. *Passing standard:* 70% with three problems essentially complete. **Justify all your answers.**

(1) Let $f\colon M \to N$ be a smooth map between manifolds. Show that

$$G = \{(p, f(p)\,;\, p \in M\} \subset M \times N$$

is a submanifold of $M \times N$ diffeomorphic to M.

(2) Let $M \subset \mathbf{R}^3$ be a right cylinder over a circle of radius R.
  (a) Find an immersion $f\colon \mathbf{R}^2 \to \mathbf{R}^3$ with image M such that the induced metric is $dx^2 + dy^2$ when written in the local coordinates $(x, y) \in \mathbf{R}^2$.
  (b) Describe all geodesics on M.
  (c) Compute the Gauss curvature of M.

(3) Let $X \subset \mathbf{R}^3$ be the surface with parameterization

$$(v \cos u, -v \sin u, bu), \quad b \neq 0, \quad (u, v) \in \mathbf{R}^2.$$

Give X the induced metric.
  (a) Compute $*du$, $*dv$, and $*(du \wedge dv)$.
  (b) Compute a local expression in the coordinates (u, v) for the Laplacian operator Δ on functions and 2-forms.

(4) (a) Define what it means for a manifold M to be *parallelizable.* and characterize parallelizability in terms of vector fields on M.
  (b) Show that if M, N are parallelizable, then so is $M \times N$.
  (c) Give a counterexample to the converse of the previous statement. [You may use the fact that not every sphere is parallelizable.]

(5) Let P be a homogeneous polynomial of degree r and suppose that

$$\frac{\partial P}{\partial x_0} = \frac{\partial P}{\partial x_1} = \cdots = \frac{\partial P}{\partial x_n} = 0$$

only at the origin. Prove that $X = \{[x_0, \ldots, x_n] \in \mathbf{P}^n : P(x) = 0\}$ is an embedded submanifold of $\mathbf{P}^n$. [You may use *Euler's theorem for homogeneous functions*: If f is homogeneous in n variables $x_1, \ldots, x_n$ of degree r, then $\sum_{i=1}^n x_i \frac{\partial f}{\partial x_i} = rf$.]

(6) Let

$$J = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \in GL_{2n}(\mathbf{R}),$$

where I_n is the $n \times n$ identity matrix, and define

$$G = \{A \in GL(2n, \mathbf{R}) \, ; \, A^T J A = J\} \, .$$

(a) Show that G is a Lie subgroup of $GL_{2n}(\mathbf{R})$.
(b) Compute its Lie algebra and its dimension.
(c) Show that $G \subset SL_n(\mathbf{R})$.
(d) Show that G acts transitively on $\mathbf{R}^{2n}$, where elements of $\mathbf{R}^{2n}$ are given by column vectors, and the action is matrix multiplication.

(7) (a) Define a *connection* on a smooth manifold.
(b) Define the *Levi-Civita connection* on a Riemannian manifold (M, g)
(c) Let (M, g) be a Riemannian manifold and let ∇ be the Levi-Civita connection. Let Φ be a covariant tensor of order r, i.e. for each open subset $U \subset M$, Φ defines a $C^\infty(U)$-multilinear map:

$$\Phi \colon \mathcal{X}(U) \times \cdots \times \mathcal{X}(U) \to C^\infty(U) \, .$$

Given $X \in \mathcal{X}(U)$ we define $\nabla_X \Phi$ by

$$(\nabla_X \Phi)(Y_1, \ldots, Y_r) := X \, \Phi(Y_1, \ldots, Y_r) - \sum_{i=1}^{r} \Phi(Y_1, \ldots, \nabla_X Y_i, \ldots, Y_r) \, .$$

Prove that $\nabla_X \Phi$ is a a covariant tensor of order r.
(d) Prove that

$$\nabla_X (f \Phi) \;=\; (X f) \Phi + f \nabla_X \Phi \, .$$

(8) Let $\varphi \in \Lambda^r(M)$ and $X_1, \ldots, X_{r+1} \in \mathcal{X}(M)$. Show that:

$$d\varphi \left(X_1, \ldots, X_{r+1}\right) = \sum_{i=1}^{r+1} (-1)^{i-1} \, X_i \varphi \left(X_1, \ldots, \hat{X}_i, \ldots X_{r+1}\right) + \sum_{i<j} (-1)^{i+j} \varphi \left([X_i, X_j], X_1, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots X_{r+1}\right)$$