Answer five of the seven questions. Indicate clearly which five questions you want graded. Justify your answers.

Passing standard: For Master’s level, 60% with two questions essentially complete. For Ph.D. level, 75% with three questions essentially complete.

(1) Show that a continuous map $S^1 \to \mathbb{R}$ cannot be either injective or surjective.

(2) Let $X$ be a topological space.
   (a) Write careful definitions of the statements “$X$ is connected”, “$X$ is path connected” and “$X$ is locally path-connected”.
   (b) Show directly from the definitions that a connected and locally path-connected space is path-connected.

(3) Let $X$ be a topological space.
   (a) Show that $X$ is Hausdorff if and only if the diagonal $\Delta = \{(x, x) \in X \times X \mid x \in X\}$ is closed in $X \times X$.
   (b) Suppose that $X$ is Hausdorff. Let $A$ be a dense subset of a space $Y$. Show that if $f, g : Y \to X$ are continuous functions that agree on $A$ (i.e. $f|_A = g|_A$), then $f = g$.

(4) Let $X$ and $Y$ be spaces, and suppose that $X$ is compact. Show directly from the definitions that the projection $\pi : X \times Y \to Y$ is a closed map.

(5) Let $X$ be a metric space, and $A \subset X$ a subspace. Recall that $X/A$ denotes the quotient space of $X$ where all the points of $A$ have been identified.
   (a) Show that $X/A$ is Hausdorff if and only if $A$ is closed.
   (b) If $X = \mathbb{R}^2$ and $A$ is the closed unit ball, show that $X/A$ is homeomorphic to $X$.

(6) Define a sequence of functions $\{f_n\}$, $f_n : \mathbb{R} \to \mathbb{R}$ by
   \[ f_n(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ e^{nx} & \text{if } x < 0. \end{cases} \]
   Determine whether or not the sequence converges in the point-open, uniform, and compact open topologies (the point-open topology is the same as the product topology, where the space of functions $\mathbb{R} \to \mathbb{R}$ is considered as a product of uncountably many copies of $\mathbb{R}$).

(7) Let $X$ be a metric space, and let $B([0,1], X)$ denote the set of bounded functions $[0,1] \to X$, endowed with the sup norm metric. Show that $B([0,1], X)$ is complete if and only if $X$ is complete.