Provide solutions for Eight of the following Ten problems. Each problem is worth 10 points. To pass at the Master’s level, it is sufficient to have 45 points, with 3 essentially correct solutions; 55 points with 4 essentially correct solutions are sufficient for passing at the Ph.D. level. Indicate clearly which problems you want graded.

NOTATION: We denote by \( \mathbb{D} \) the open unit disc, i.e. \( \mathbb{D} = \{ z \in \mathbb{C} \mid |z| < 1 \} \).

1. Use contour integration to verify that for \( b > 0 \),
\[
\int_{-\infty}^{\infty} \frac{\cos x}{x^2 + b^2} \, dx = \frac{\pi e^{-b}}{b}.
\]
Be sure to justify all your steps.

2. Prove that for any \( a > 1 \), the equation \( z = e^{z-a} \) has exactly one solution in the unit disc \( \mathbb{D} \). (Give the full statement of any theorem you use).

3. Locate the poles of
\[
f(z) = \frac{\tan(z)}{z^5},
\]
and indicate the order of each pole. Find the principal part, i.e. the coefficients of the negative powers, in the Laurent expansion of \( f \) at each pole.

4. (a) State Morera’s theorem.
(b) Use Morera’s Theorem to prove that if \( f \) is continuous on \( \mathbb{C} \) and holomorphic on the set \( \Omega = \{ z \in \mathbb{C} \mid \text{Im}(z) \neq 0 \} \), then \( f \) is holomorphic on \( \mathbb{C} \).

5. (a) State the Schwarz Lemma, then prove it.
(b) Suppose \( f \) is a holomorphic mapping of the unit disc \( \mathbb{D} \) to itself and that \( f \) is not the identity map. Use the Schwarz lemma to prove that \( f \) has at most one fixed point in \( \mathbb{D} \).
6. For each part of this problem, indicate whether the statement is true or false. If true, give a proof; if false, provide a counterexample.
   (a) There exists a holomorphic function $f$ on the unit disc $\mathbb{D}$ such that $f(1/n) = f(-1/n) = 1/n^3$ for $n = 2, 3, \ldots$.
   (b) There exists a holomorphic function $f$ on the punctured unit disc ($\mathbb{D} - \{0\}$) such that $g(z) = e^{f(z)}$ has a simple pole at the origin.
   (c) If $f$ is a holomorphic function on the unit disc $\mathbb{D}$ which does not vanish at any point of $\mathbb{D}$, then there exists a function $g$ holomorphic on $\mathbb{D}$ satisfying $g^2 = f$. (i.e. every non-vanishing holomorphic function on $\mathbb{D}$ has a holomorphic square root on $\mathbb{D}$.)

7. Write down a conformal map that takes the “right-half” of the unit disc $R = \{z \in \mathbb{D} \mid \text{Re}(z) > 0\}$ onto the unit disc $\mathbb{D}$.

8. Use contour integration to prove that
   \[ \int_0^{\infty} \frac{x^{1/3}}{1 + x^2} \, dx = \frac{\pi}{\sqrt{3}}. \]
   Be sure to justify all your steps.

9. Evaluate
   \[ \frac{1}{2\pi i} \int_C \frac{\cos^n(z)}{z^3} \, dz \]
   where $n \geq 0$ is a non-negative integer, and $C$ is the unit circle $|z| = 1$ traversed counterclockwise once.

10. (a) Give a careful statement of the Cauchy Inequalities, then prove them by using the Cauchy Integral Formulas.
    (b) State Liouville’s theorem. Use the Cauchy inequalities to prove Liouville’s theorem.