Do 5 out of the following 7 questions. Indicate clearly which questions you want to have graded. Passing standard: 70% with three problems essentially complete. Justify all your answers.

Problem 1. Let $M$ be a manifold, $f : M \to \mathbb{R}$ be a smooth map and $p \in M$ a critical point of $f$.
   
a) Define what is meant by the Hessian of $f$ at $p$. If your definition involves choices, show that the end-result is independent of those choices.
   
b) Let $f : S^n \to \mathbb{R}$ be the map
   \[ f(x) = \sum_{j=1}^{n+1} j x_j^2; \quad x = (x_1, \ldots, x_{n+1}) \in S^n \subset \mathbb{R}^{n+1}. \]
   
   Find the critical points of $F$.
   
c) Compute the Hessian, Hess($f$), at each of the critical points obtained above.

Problem 2. Let $H = \{(x, y) \in \mathbb{R}^2 : y > 0\}$ with the Lie group structure defined by the product:
   \[(x, y) \ast (x', y') = (x + x', yy').\]
   
a) Find a basis of left-invariant 1-forms on $H$ and define a left-invariant metric on $H$.
   
b) Compute the Gaussian curvature of the metric defined in a).
   
c) Compute the geodesics of $H$ with respect to the metric defined in a).

Problem 3. Let $f : \mathbb{RP}^1 \to \mathbb{RP}^n$ be defined by
   \[ [s : t] \mapsto [s^n : s^{n-1}t : \cdots : st^{n-1} : t^n]. \]
   
a) Show that $f$ defines an embedding of $\mathbb{RP}^1$ in $\mathbb{RP}^n$.
   
b) Let $T_n \to \mathbb{RP}^n$ be the tautological line bundle and $1_n \to \mathbb{RP}^n$ the trivial line bundle over $\mathbb{RP}^n$. What is the relationship between $f^*(T_n)$ and the bundles $T_1$ and $1_1$ over $\mathbb{RP}^1$?
Problem 4. Let \((M, g)\) be an oriented Riemannian manifold and let \(\Omega\) denote its volume element.

a) Define the Laplace operator \(\triangle\) on smooth functions \(f \in C^\infty(M, \mathbb{R})\) on \(M\).

b) If \(M\) is compact and \(f, h : M \to \mathbb{R}\) are smooth functions then prove that
\[
\int_M h(\triangle f) \Omega = -\int_M g(df, dh) \Omega
\]

c) Show that if \(M\) is compact, every function with \(\triangle f = 0\) is necessarily constant.

Problem 5. Let \(G\) be a compact Lie group and \(\sigma : G \to G\) a Lie group homomorphism such that \(\sigma \circ \sigma = \text{id}\).

a) Prove that \(H = \{ g : \sigma(g) = g \}\) is a compact Lie subgroup of \(G\).

b) Show that \(G\) has a bi-invariant Riemannian metric relative to which \(\sigma\) is an isometry.

Problem 6. Consider the 1-forms in \(\mathbb{R}^4\):

\[
\alpha = (x_1^2 - x_2^2) \, dx_1 - 2x_1 x_2 \, dx_2 + dx_3 ; \quad \beta = 2x_1 x_2 \, dx_1 + (x_1^2 - x_2^2) \, dx_2 + dx_4
\]

For each \(p \in \mathbb{R}^4\), let
\[
\Delta(p) := \{ v \in T_p(\mathbb{R}^4) : \alpha(p)(v) = \beta(p)(v) = 0 \}
\]

a) Show that \(\Delta\) is a 2-dimensional involutive distribution in \(\mathbb{R}^4\).

b) Given \(p = (0, 0, a, b) \in \mathbb{R}^4\), construct a function \(F : U \to \mathbb{R}^2\), where \(U \subset \mathbb{R}^2\) is an open neighborhood of the origin, such that
   (a) \(F(0, 0) = (a, b)\)
   (b) \(\text{Graph}(F) \subset \mathbb{R}^4\) is an integral submanifold of \(\Delta\).

Problem 7. We identify \(S^2\) with the Riemann sphere \(\mathbb{C} \cup \{\infty\}\) and define \(F : S^2 \to S^2\) by \(F(z) = z^3\) if \(z \in \mathbb{C}\), and \(F(\infty) = \infty\).

a) Prove that \(F\) is a \(C^\infty\) map.

b) Show that for all \(\omega \in \Lambda^2(S^2)\),
\[
\int_{S^2} F^*(\omega) = 3 \int_{S^2} \omega
\]