NAME:

Advanced Analysis Qualifying Examination
Department of Mathematics and Statistics
University of Massachusetts

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Instructions

1. This exam consists of eight (8) problems all counted equally for a total of 100%.
2. You are encouraged to try to solve every problem; there is no penalty for incorrect answers.
3. In order to pass this exam, it is enough that you solve essentially correctly at least five (5) problems and that you have an overall score of at least 65%.
4. State explicitly all results that you use in your proofs and verify that these results apply.
5. Please write your work and answers clearly in the blank space under each question.

Conventions

1. For a set A, 1_A denotes the indicator function or characteristic function of A.
2. If a measure is not specified, use Lebesgue measure on $I\!R$. This measure is denoted by m.
3. If a σ-algebra on $I\!R$ is not specified, use the Borel σ-algebra.

1. In each case compute $\int_X f\, d\mu$, where $X = \{1, 2, 3, \ldots\} = I\!N$ and all subsets of X are measurable.

   (a) $f(x) = 1/x$ and μ is counting measure; i.e., $\mu(B)$ equals the number of elements, finite or infinite, in B.

   (b) $f(x) = 2^{-x}$ and μ is counting measure; i.e., $\mu(B)$ equals the number of elements, finite or infinite, in B.

   (c) $f(x) = e^x$ and $\mu(B) = 1_B(13)$.

   (d) $f(x) = (x\pi^x)^{-1}$ and $\mu(\{k\}) = k(\pi/2)^k$ for all $k \in X$.

2. Let $\{f_j, j \in I\!N\}$ be a sequence of functions in $L^2[0,1]$ with Fourier coefficients

$$\hat{f}_j(n) = \int_0^1 e^{-2\pi i n x} f_j(x) dx.$$

   Assume the following about $\{\hat{f}_j(n)\}$.

(a) There exists a positive sequence in $\ell^2(\mathbb{Z})$, $\{c(n), n \in \mathbb{Z}\}$, such that

$$\sup_{j \in \mathbb{N}} |\hat{f}_j(n)| \le c(n) \text{ for all } n \in \mathbb{Z}.$$

(b) For each fixed $n \in \mathbb{Z}$, $\hat{f}_\infty(n) = \lim_{j\to\infty} \hat{f}_j(n)$ exists.

Prove that there exists a function $f \in L^2[0,1]$ such that $f_j \to f$ in $L^2[0,1]$.

3. Let $C[-1,1]$ denote the space of bounded continuous functions mapping $[-1,1]$ into $\mathbb{R}$.

(a) Prove that $C[-1,1]$ is a Banach space with norm $\|f\| = \sup_{t\in[-1,1]} |f(t)|$.

(b) Let T be a bounded linear operator mapping $C[-1,1]$ into $\mathbb{R}$. Give the definition of the operator norm $\|T\|$.

(c) For $f \in C[-1,1]$ define

$$Sf = \int_{-1}^{1} x^2 f(x)dx \text{ and } Tf = \int_{-1}^{1} x^3 f(x)dx.$$

(i) Give the numerical values of $\|S\|$ and $\|T\|$.

(ii) Is the supremum in the definition of $\|S\|$ attained? Is the supremum in the definition of $\|T\|$ attained? Explain your answers.

4. The purpose of this problem is to prove that

$$\lim_{n\to\infty} \sqrt{n} \int_0^{\pi/2} \cos^n(x)\, dx = \int_0^\infty \exp[-x^2/2]\, dx.$$

Prove this by combining the limit in part (a) and the steps in part (b).

(a) First show that for any $0 < \delta < \pi/2$

$$\lim_{n\to\infty} \sqrt{n} \int_\delta^{\pi/2} \cos^n(x)\, dx = 0.$$

(b) For any $0 < \delta < \pi/2$ and $n \in \mathbb{N}$ the change of variable $y = \sqrt{n}x$ gives

$$\sqrt{n} \int_0^\delta \cos^n(x)\, dx = \int_0^{\delta\sqrt{n}} \cos^n(y/\sqrt{n})\, dy.$$

Complete the proof of the limit in the first display in this problem by using without proof the inequality

$$1 - \frac{1}{2}x^2 \le \cos(x) \le \exp[-x^2/2] \text{ for } 0 \le x \le \pi/2.$$

5. Let $(X, \mathcal{M})$ be a measure space. Let $\{f_n, n \in \mathbb{N}\}$ be a sequence of measurable functions mapping X into $\mathbb{R}$ and let f be a measurable function mapping X into $\mathbb{R}$.

(a) Define the set

$$A = \{x \in X : \lim_{n\to\infty} f_n(x) = f(x)\}.$$

Prove that $A \in \mathcal{M}$.

(b) Define the set

$$B = \{x \in X : \{f_n(x), n \in I\!N\} \text{ is not a Cauchy sequence}\}.$$

Prove that $B \in \mathcal{M}$.

6. Let $f : I\!R \to I\!R$ be a nondecreasing function. It is known that f is differentiable a.e. and that f' is a measurable function. Prove that for any closed bounded interval $[a, b]$

$$\int_a^b f'(x)\, dx \;\le\; f(b) - f(a).$$

Hints. Work with the function $h(x)$ which equals $f(x)$ for $x \in [a, b]$ and equals $f(b)$ for $x > b$. Apply Fatou's lemma.

7. Let $(X, \mathcal{M}, \mu)$ be a measure space, $\{f_n, n \in I\!N\}$ a sequence of nonnegative $L^2(\mu)$-functions mapping X into $[0, \infty)$, and f a nonnegative $L^2(\mu)$-function mapping X into $[0, \infty)$. Denote the $L^2(\mu)$-norm by $\|\cdot\|$. Assume that

$$f_n \to f \text{ a.e.} \quad \text{and} \quad \lim_{n\to\infty} \|f_n\| = \|f\|.$$

Prove that $\lim_{n\to\infty} \|f - f_n\| = 0$. (**Hint.** Expand $\|f - f_n\|^2$.)

8. Let $(X, \mathcal{M}, \mu)$ and $(Y, \mathcal{N}, \nu)$ be measure spaces that are **not** σ-finite.

(a) Let A be a set in $\mathcal{M}$ and B a set in $\mathcal{N}$. For $x \in X$ and $y \in Y$ define the functions $f(x) = 1_A(x)$, $g(y) = 1_B(y)$, and $h(x, y) = f(x)g(y)$. Without using the Fubini-Tonelli Theorem, prove that

$$\int_{X\times Y} h\, d(\mu \times \nu) \;=\; \int_X f d\mu \cdot \int_Y g d\nu.$$

(b) Let f be a nonnegative measurable function mapping X into $[0, \infty)$ and g a nonnegative measurable function mapping Y into $[0, \infty)$. For $x \in X$ and $y \in Y$ define $h(x, y) = f(x)g(y)$. Without using the Fubini-Tonelli Theorem, prove that

$$\int_{X\times Y} h\, d(\mu \times \nu) \;=\; \int_X f d\mu \cdot \int_Y g d\nu.$$

(c) Explain why in parts (a) and (b) the Fubini-Tonelli Theorem cannot be applied.